Niveau(x) d'enseignement: 4 Numéro EDUQ : 0870385
Centre d'intérêt: 841

Matte, Jean.

Mathématique : second cycle : fascicule A : arithmétique et algèbre / [Jean Matte, Jean-Berchmans Veilleux] ; Direction générale des programmes, Direction de la formation générale. -- [Québec] : Ministère de l'éducation, c1986.
215 p. : ill. ; 22 x 28 cm. -- (Guide pédagogique / Ministère de l'éducation du Québec, Direction générale des programmes. Secondaire)

Version anglaise publ. ss le titre : Mathematics : second cycle : booklet A : arithmetic and algebra. Code 1633-02-01A. Code: 16-3302-01. Bibliogr.: p. 212-215.

Vedette(s)-secondaire(s): Veilleux, Jean-Berchmans • Québec (Province) Ministère de l'éducation. Direction de la formation générale

---

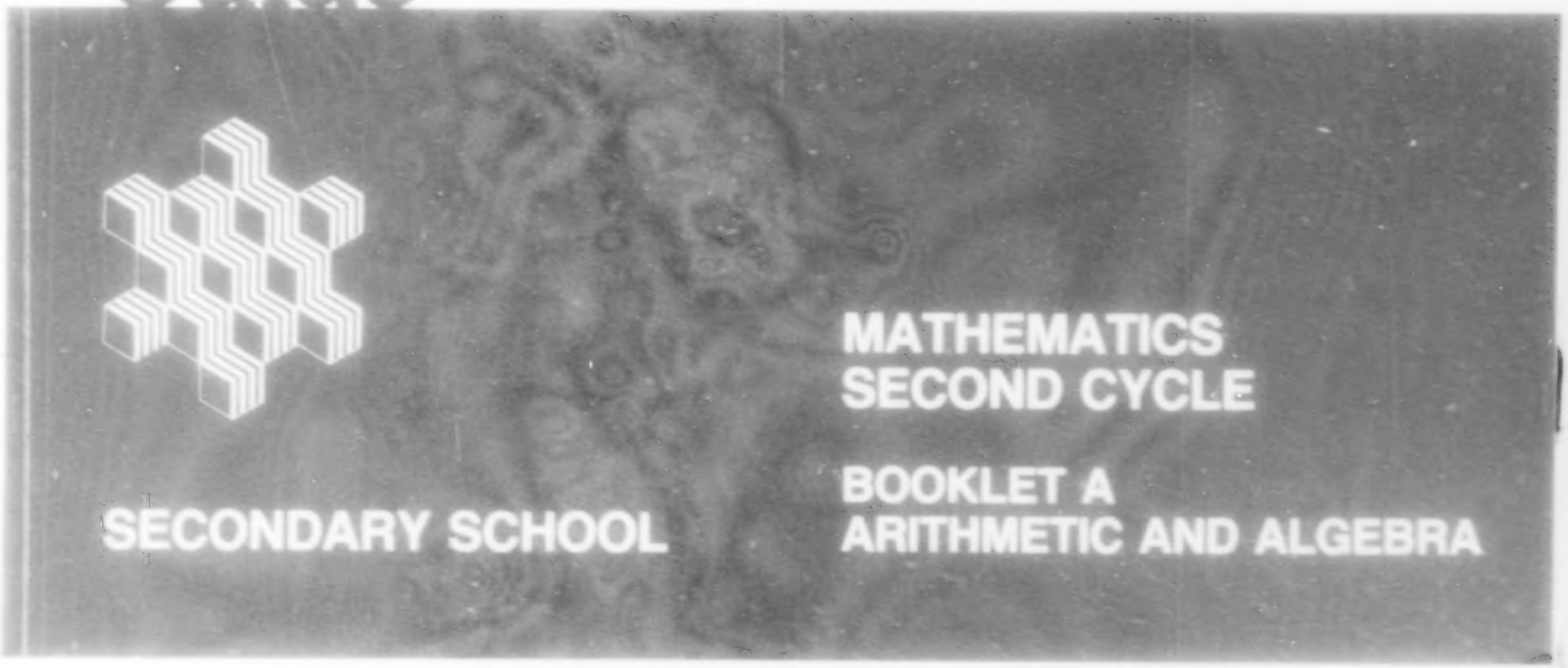
Descripteur(s): Mathématiques • Géométrie • Guide pédagogique • Secondaire (Ordre d'enseignement)

Le guide pédagogique est un complément au programme de mathématiques pour le second cycle du secondaire. Son contenu n'est pas obligatoire, et le document est présenté comme une aide pour l'enseignement. Son objectif général est de favoriser une meilleure compréhension du programme, de suggérer des approches pédagogiques et des activités d'apprentissage adaptées à l'implantation du programme. Il vise aussi à encourager les enseignants à être créateurs dans leurs techniques d'enseignement. Le guide contient un certain nombre de notes sur les objectifs généraux, terminaux et intermédiaires. Leur premier objectif est de définir le but d'un cours, de suggérer des approches et de proposer des stratégies. Le chapitre sur l'évaluation propose deux modèles d'évaluation: l'un sur l'évaluation formative et l'autre sur l'évaluation sommative. Le chapitre suivant indique le matériel didactique nécessaire pour le cours et le dernier chapitre présente une bibliographie sélective.

Type de document: d

ERIC 88/0385

Curriculum Guide



SECONDARY SCHOOL

MATHEMATICS
SECOND CYCLE

BOOKLET A
ARITHMETIC AND ALGEBRA

Québec

EDUQ 8870383

Curriculum Guide

SECONDARY SCHOOL

MATHEMATICS
SECOND CYCLE

BOOKLET A
ARITHMETIC AND ALGEBRA

Direction générale des programmes
Direction de la formation générale

Approved by the Comité catholique and
the Comité protestant of the Conseil supérieur
de l'éducation on November 4, 17 and 18, 1983.

© Gouvernement du Québec
Ministère de l'Éducation, 1986

ISBN 2-550-13507-5

Legal Deposit — Fourth Quarter 1986
Bibliothèque nationale du Québec

AUTHORS*

Jean Matte	Educational Development Officer, in charge of the Secondary School Mathematics programs, Direction générale du développement pédagogique, ministère de l'Éducation.
Jean-Berchmans Vellieux	Educational Development Officer, Direction générale du développement pédagogique, ministère de l'Éducation.

ADVISORY COMMITTEE*

Bernard Blain	Educational Development Officer, Direction générale du développement pédagogique, ministère de l'Éducation.
Noëlange Boisclair	Coordinator, Coordination provinciale en mathématique au collégial.
Ray Clarke	Q.A.M.T. (Québec Association of Mathematics Teachers) Representative.
Claude Janvier	Professor, Université du Québec à Montréal.
Raymond Lalonde	A.M.Q. (Association mathématique du Québec) Representative.
Jean Matte	Educational Development Officer, in charge of the Secondary School Mathematics programs, Direction générale du développement pédagogique, ministère de l'Éducation.
Jean-Marcel Mius d'Entremont	G.R.M.S. (Groupe des responsables en mathématique au secondaire) Representative.
Jean-Berchmans Vellieux	Educational Development Officer, Direction générale du développement pédagogique, ministère de l'Éducation.
Michel Warisse	G.R.M.S. (Groupe des responsables en mathématique au secondaire) Representative.

---

* The persons whose names are listed here were holding their respective posts during the 1981-1982 school year.

PRODUCTION COMMITTEE*

Michel Arconet	Teacher, C.S.R. Meilleur.
Emilienne Boisvert	Teacher, C.S.R. Jérôme-Le Royer.
Marcel Brisebois	Teacher, C.E.C.M. (Commission des écoles catholiques de Montréal).
Michael Cassidy	Educational Consultant, C.S. Baldwin-Cartier.
Raymond Lalonde	Teacher, C.E.C.M. (Commission des écoles catholiques de Montréal).
Jean Mattei	Educational Development Officer, in charge of the Secondary School Mathematics programs, Direction générale du développement pédagogique, ministère de l'Éducation.
Jean-Marcel Mius d'Entremont	General Education Coordinator, C.S.R. Tilly.
Pierre-Paul Renaud	Educational Consultant, C.S.R. Blainville-Deux-Montagnes.
Jean-Gerchmans Veilleux	Educational Consultant, C.S.R. Chambly.
Michel Warisse	Teacher, C.S. Châteauguay.
Guy Zanaitis	Teacher, C.E.C.M. (Commision des écoles catholiques de Montréal).
Edward Zegray	Educational Consultant, C.E.C.M. (Commission des écoles catholiques de Montréal).

TRANSLATION

Asselin Charles	Translator, with the assistance of the members of the Educational Development in the English Language Team, ministère de l'Éducation.

---

* The persons whose names are listed above were holding their respective posts during the 1981-1982 school year.

TABLE OF CONTENTS

PAGE

1. INTRODUCTION 1

2. TEACHING NOTES 7

- General Objective 1 9
  - Terminal Objective 1.1 17
  - Terminal Objective 1.2 37
  - Terminal Objective 1.3 53
  - Terminal Objective 1.4 85
  - Terminal Objective 1.5 101
  - Supplementary Teaching Notes 132

- General Objective 2 ..... Booklet B

- General Objective 3 ..... Booklet C

3. EDUCATIONAL EVALUATION 187

4. INSTRUCTIONAL MATERIALS 201

5. BIBLIOGRAPHY 209

1. INTRODUCTION

This curriculum guide[1] is a supplement to the Mathematics Program for the second cycle of secondary school (Document No. 16-3302-A). Its content is not compulsory and the document itself is being suggested as an optional teaching aid.

> "Its general intention is to promote a better understanding of the program and to suggest certain teaching approaches and learning activities which are considered suitable for an effective implementation of the program. It also aims to encourage teachers to be creative in their search for pertinent and effective teaching techniques."[2]

This guide contains a number of notes on each of the general, terminal, and intermediate objectives. These notes and commentaries are useful guidelines for planning courses. Their first objective is to define scope, to suggest approaches, and to propose strategies. They comply with the major principles of methodology and adhere to the guiding principles which governed the development of the program.

In several places, the curriculum guide suggests a number of enrichment activities that are likely to enable motivated pupils to explore more thoroughly certain areas covered by the program.

---

1. The curriculum guide is made up of three booklets:
   - Booklet A: Arithmetic and Algebra (Document 16-3302-01A);
   - Booklet B: Geometry (Document 16-3302-02A);
   - Booklet C: Probability and Statistics (Document 16-3302-03A).

2. D.G.D.P., Cadre relatif à l'élaboration des programmes et des guides pédagogiques, p. 19. This citation is a free translation.

Teachers should adopt instructional methods that enable them to make effective use of technology in the classroom. One of the guiding principles underlying this program encourages the use of such instruments as the calculator and the microcomputer. There are several subjects covered in the program for which the use of these instruments is particularly appropriate. These instruments are useful, for example, for:

- finding number patterns;
- evaluating a sequence of operations on large and small numbers;
- solving equations by means of iterative methods;
- solving complex problems;
- validating approximations;
- verifying answers;
- evaluating combinative and statistical formulae;
- defining trigonometric ratios.

The last two subjects listed above actually require the use of electronic instruments. Without these instruments, in fact, evaluating combinative or statistical formulae would be an excessively difficult process. Moreover, as the trigonometric tables are not used, the calculator must be employed. To the extent that the pupil is encouraged to use a variety of algorithms for solving realistic and, therefore, very complex problems, the use of electronic instruments should free him from purely operational concerns. Consequently, he should be able to concentrate on selecting and applying the most effective strategies.

It should also be stressed that the curriculum guide is not a "lesson plan" and that the information it contains should not be transmitted directly to the pupil. The notes and commentaries are intended for the teacher, not for the pupil. It is to be expected, therefore, that some materials are treated in greater detail in this guide than in those texts intended for presentation to pupils.

It is essential to ensure that all pupils have a good grasp of this basic program. It is equally important to provide remedial work for the weaker pupils, and enrichment activities based on pertinent materials for the more advanced pupils. It is the responsibility of the schools to define the content of the remedial and enrichment activities.

The use of problem-solving techniques is stressed throughout this guide so as to highlight the practical aspect of mathematics and to unify the different subject areas developed in the program (arithmetic, algebra, geometry, trigonometry, statistics and probability). Also, in the context of a basic mathematics program the importance of cultivating operational skills should not be underestimated. A good understanding of the basic concepts does not make the acquisition of reliable and effective algorithms superfluous.

Far from neglecting the contribution of the new technologies, it is important to consider carefully the impact of the pocket calculator on society, particularly in the schools. Many pupils own one and want to use it.

The chapter on educational evaluation suggests a model, Form A, designed to facilitate the continuous evaluation of the pupil's learning. Such an evaluation enables the teacher to adapt his remedial teaching activities to the particular needs of each pupil. Another form, Form B, could eventually serve as an evaluation instrument which could help school authorities to make decisions respecting the promotion of each pupil.

For additional information, the teacher should consult Chapters 4 and 5 ("Instructional Materials" and "Bibliography"). The information given in these two chapters is far from exhaustive and may be supplemented by an inventory of local resources.

In order to standardize, and ensure uniform interpretation of, the various graphs, notations, and symbols, the teacher should consult the reference manual entitled Information Document--Graphs, Notations, and Symbols Used in Secondary Mathematics (Code 16-3306A). The information and guidelines found in this document will be used in the tests or exams prepared by the ministère de l'Éducation.

2. TEACHING NOTES

GENERAL OBJECTIVE

1

TO ENCOURAGE pupils to apply their knowledge of arithmetic and algebra.

SUBJECT AREAS: Arithmetic and Algebra

TIME: 30% in Secondary III

30% in Secondary IV

35% in Secondary V

The underlying concern in this first general objective is to enable the pupil to develop the basic knowledge of arithmetic and algebra which is indispensable for functioning in today's society. Such knowledge should enable the pupil to use mathematics routinely in everyday living, and to realize how, by means of "reduced models," it is possible to solve a given situation mathematically by:

- "mathematizing" it (constructing a mathematical model),
- operating on the model,
- and interpreting the results.

It is important to help the pupil develop the skills needed for mathematizing, operating, and interpreting, and at the same time to teach him the algorithms used for solving problems.

It should be obvious that mathematics can play an important role in a broad variety of fields of activity, for example in:

- consumption,
- finance,
- taxation,
- sociology,
- computer science,
- social studies,
- pure sciences.

It is important, therefore, to stress the social impact of mathematics.

The study of equations or inequations in one or more variables has a broad variety of applications. Problems which require the use of inequations call for optimal solutions. In this context, linear programming seems to be the ultimate application of the study of equations and inequations to the extent that these are used for creating a model by means of which an actual situation can be better understood and interpreted.

In summary, the learning content of this objective should enable the pupil to master the concepts and algorithms that are necessary for a better interpretation of his surroundings, and to develop a number of skills pertaining to mathematization, communication, and application of concepts and principles.

The pupil will acquire a more solid knowledge of arithmetic and algebra if instructional methods are adapted to his age, interests, and life experience. It is important, therefore, to develop problem-solving approaches and strategies in order to highlight the useful and practical aspects of mathematics.

It should be pointed out again that, in a broad sense, there are four main stages in the problem-solving process:

- understanding the problem;
- developing a plan to solve the problem;
- executing the plan;
- analyzing the solution.

These main stages encompass a number of secondary steps which are all geared to the solution of the problem involved.

This is a partial but comprehensive list of strategies applicable to each of these stages.

1. Understanding the problem:

   - Read the problem several times and make sure that each word is understood;
   - Reformulate the problem in your own words;

- Identify the data;
- Identify what is being sought;
- Illustrate the data.

2. Developing a plan to solve the problem:

- Eliminate non-pertinent data;
- Find the missing data;
- Construct or complete a figure;
- Describe the principal conditions which the result should meet;
- Subdivide the problem into smaller problems;
- List the criteria on which to base your choices if the problem involves decision making;
- Find one or several statements, one or several formulas, one or several properties, etc.;
- Formulate hypotheses;
- Do a simulation of the problem;
- Find a model;
- Backtrack;
- Proceed by the process of elimination;
- Determine the appropriate operation(s) in a given order;
- Select the most suitable tools, procedures, algorithms;
- Make use of various available resources.

3. Executing the plan:

- Evaluate if the solution is possible;
- Perform the required calculations;
- Recall a concept, a formula, an algorithm;
- Use a table, a calculator, etc.

4. Analyzing the solution:

- Check the correctness of the results;
- Check the calculations and the procedure followed;
- Look for other possible procedures;
- Provide complete information in the answer;
- Create and solve similar problems.

When applied to the solution of written problems, the strategies described above could be summed up in the following way:

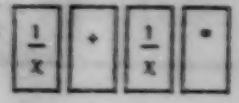
1- Read the problem carefully;
2- Restate the problem in your own words;
3- Select the pertinent data in the problem;
4- Mathematize the situation;
5- Estimate the result;
6- Solve the problem;
7- Interpret the result obtained.

To solve any problem it is necessary to use a number of strategies drawn from the preceding list.

It is essential that pupils be thoroughly familiar with the rules of algebraic operations, and have a clear understanding of the reasoning behind them. Pupils should be encouraged to use the calculator to check the accuracy of the results of operations performed on numbers. They should not, however, become dependent on the calculator. Whenever the pupil uses the calculator he should always estimate, verify, and look for sequences or procedures as well.

The calculator should also be used for exploring number patterns, for finding constants, for solving an equation or a system of equations, for verifying the estimated result of an operation, and so on.

In this curriculum guide, a sequence refers to an ordered series of actions which involve pressing on a certain number of keys on the calculator. For example, $\frac{3 - 11}{7}$ can be translated by the sequence [(] 3 [-] 11 [)] [÷] 7 [=]. A procedure is a sequence that is independent from the value that is computed. For example, the procedure [√] [-] 7 [=] can be associated with $\sqrt{x} - 7$, and if this procedure is applied to the number 25, the result is -2.

The teacher should be aware that the semantics of sequences and procedures can be confusing to the pupil and can affect his understanding of the expressions under consideration. For example, the procedure  gives 2½ if the initial value is 2, whereas the expression $\frac{1}{X} + \frac{1}{X}$ gives 1 if X = 2.

NOTE

(a) Calculators

The sequences or procedures suggested in the guide are valid only on certain types of calculators. They should, therefore, be used with some caution. However, generally, with some minor adjustments, they can be made to fit the particular features of any given calculator model.

(b) Computers

The examples of programs included in the guide are in BASIC. They may, therefore, require some minor modifications of their syntax before they can be used on a particular computer.

TERMINAL OBJECTIVE 1.1 TO SOLVE real-life problems involving real numbers.

In the first cycle of secondary school, the pupil studies the natural numbers, the whole numbers, and the rational numbers. In this initial part of the second cycle program, he will be completing his study of numbers by exploring the real numbers. Through intuition and observation, the pupil should be able to discover the characteristics of these numbers.

The calculator can be a very useful learning aid, in the following cases, for example. The rules of signs can be inferred by observing patterns in operations involving addition and multiplication of positive and negative numbers. By considering the quotient of two pertinently selected integers, the pupil can discover the cycles of decimal development, that is, the rational numbers. The square root of numbers that are not perfect squares never gives periodic numbers; these are non-periodic or irrational numbers. The last two cases provide an opportunity to introduce scientific notation and to develop the ability to round numbers and estimate operation results.

Real-life problems often involve ratios and proportions. The pupil should, therefore, develop a solid understanding of these concepts if he is to be able to give a better interpretation of real-life situations.

The following is a list of typical situations involving real numbers:

- calculating profits and losses;
- finding the perimeter of a figure, the area of a surface, or the volume of a solid;
- locating an object on a line, on a plane, or in space;
- using ratios to represent an object to scale;
- etc.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.1 To IDENTIFY the rules of signs, given sign patterns.	III	- The pupil will not have completed his study of the rules of signs for multiplication and division in the first cycle of secondary school. With this objective, he has the opportunity to do so while at the same time strengthening his grasp of the subject. - The pupil might be asked to complete the following tables by filling out the blanks in each line or column. This task involves observing and following certain patterns among the number sequences.

+	...	-3	-2	-1	0	1	2	3	...
...	...	..	..	..	..	..	..	..	...
3	...	0	1	2	3	4	5	6	...
2	...	-1	0	1	2	3	4	5	...
1	...	-2	-1	0	1	2	3	4	...
0	...	-3	-2	-1	0	1	2	3	...
-1	...	-4	-3	-2	-1	0	1	2	...
-2	...	-5	-4	-3	-2	-1	0	1	...
-3	...	-6	-5	-4	-3	-2	-1	0	...
...	...	..	..	..	..	..	..	..	...

x	...	-3	-2	-1	0	1	2	3	...
...	...	..	..	..	..	..	..	..	...
3	...	-9	-6	-3	0	3	6	9	...
2	...	-6	-4	-2	0	2	4	6	...
1	...	-3	-2	-1	0	1	2	3	...
0	...	0	0	0	0	0	0	0	...
-1	...	3	2	1	0	-1	-2	-3	...
-2	...	6	4	2	0	-2	-4	-6	...
-3	...	9	6	3	0	-3	-6	-9	...
...	...	..	..	..	..	..	..	..	...

- If the pupil examines the two completed tables carefully, he should be able to infer the rules of signs from the patterns that can be identified.

+	...	-3	-2	-1	0	1	2	3	...
...	...	..	..	..	..	..	..	..	...
3	...	0	1	2	3	4	5	6	...
2	...		0	1	2	3	4	5	...
1	...			0	1	2	3	4	...
0	...				0	1	2	3	...
-1	...					0	1	2	...
-2	...						0	1	...
-3	...							0	...
...	...	..	..	..	..	..	..	..	...

x	...	-3	-2	-1	0	1	2	3	...
...	...	..	..	..	..	..	..	..	...
3	...				0	3	6	9	...
2	...				0	2	4	6	...
1	...				0	1	2	3	...
0	...	0	0	0	0	0	0	0	...
-1	...				0				...
-2	...				0				...
-3	...				0				...
...	...	..	..	..	..	..	..	..	...

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.1 To IDENTIFY the rules of signs, given sign patterns (cont'd).	III	For addition, the sum of two numbers: - has the same sign as the number with the greatest absolute value; - is equal in absolute value: . to the sum of the absolute values of the two numbers, if these numbers have the same sign, . to the absolute value of the differences between the absolute values of the two numbers, if these numbers have opposite signs. For multiplication, the product of two numbers: - is positive, if the two numbers have the same sign; - is negative, if the two numbers have opposite signs; - is equal in absolute value to the product of the absolute values of the two numbers. - It is advisable to define absolute value for the pupil by means of such examples as $\|8\| = 8$, $\|-3\| = 3$, and so on. This should be done informally, and it is not necessary to study the properties of absolute value. - Formally, absolute value can be defined as follows: $\forall a \in \mathbb{R}_+ : \|a\| = a$ $\forall a \in \mathbb{R}_- : \|a\| = -a$ - The teacher might also encourage the pupil to find other interesting patterns by examining the different diagonal sequences. - The preceding tables could be constructed on the computer. The teacher might suggest that the pupil write a program and run it in the computer. The following is an example of a BASIC program for constructing the table for multiplication. Values vary from -5 to +5. 10 CLS 20 PRINT "Multiplication Signs Table." 30 PRINT 40 PRINT " ";

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.1 To IDENTIFY the rules of signs, given sign patterns (cont'd).	III	50 FOR X=-5 TO 5 60 PRINT X; 70 NEXT X 80 PRINT 90 PRINT 100 FOR Y=5 TO -5 STEP -1 110 PRINTUSING "#########";Y; 120 PRINT " "; 130 FOR X=-5 TO 5 140 IF X*Y>0 THEN PRINT "+"; ELSE IF X*Y<0 THEN PRINT "-"; ELSE PRINT "0"; 150 NEXT X 160 PRINT 170 NEXT Y 180 END - In the field of mathematics, a distinction is made between the minus sign "-" and the signs of the opposite of a number "-". Generally, the following keys on the calculator are used respectively for each of the two signs: [-] and [+/-]. The command [-] is used with two operands (diadic command) and the command [=]. The latter command triggers the execution of the subtraction. The command [+/-] makes it possible to modify the sign of the number on display. This command is known as a monadic or one-operand command. The calculator can be used to operate on large and small numbers so that the pupil can verify that the rules of signs apply to all numbers. - With such situations as those involving "profits and losses," "debits and credits," "temperatures," "relative positions of objects on a map or a plan," and so on, the pupil can more effectively identify certain rules of signs. However, since it is difficult to find a concrete situation which can be represented by (-9) + (-3), it is advisable to have the pupil construct a number table by means of which the appropriate patterns can be identified. In this particular case, the calculator makes it possible to generalize. Without it, in fact, such a generalization might seem unjustified to the pupil.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.2 To APPLY the rules of signs in the evaluation of numerical or literal expressions.	III	- Simple numerical and literal expressions should be evaluated without the help of the calculator. - The pupil should be able to determine the sign of such expressions as: - 2,75 + 7,43 - 6,994 + ... - 2,43 (3,7 - 11,71) - 6,43 ÷ (4,05) or $3ab - 4b$ if a = -0,47 and b = 3 $\frac{2at - u}{3b}$ if a = -2, b = 3,5, t = -0,7, u = -5 - Once the pupil has mastered the rules of signs, he should be able to handle problems that can be solved by means of certain formulae. Example: A group of pupils drop a ball from the roof of a building. The formula that can be used for computing the speed of the ball is $v = v_0 + at$ (neglecting air resistance), where: v is the speed of the ball, v_0 is the initial speed of the ball, a is the acceleration due to gravity a ≈ 9,806 65 m/s^2 t is the time expressed in second(s) What is the speed of the ball as it reaches the ground? Based on the results of calculations made by the pupils (three calculations per pupil), have them complete the following table using: v = 0 + 9,806 65 t = 9,806 65 t

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.2 To APPLY the rules of signs in the evaluation of numerical or literal expressions (cont'd).	III	

Time			Time Average	Speed
3,1	2,9	3,0	3,0	29,4
2,9	2,8	3,1	2,9	28,4

DA			TA	SA

Each pupil calculates the time average for each drop of the ball. The results are rounded to the nearest tenth. Then, each pupil calculates the speed, using the procedure [×] 9,806 65 [=], and the results are rounded to the nearest tenth.
Next, the pupil calculates:

- the combined time average for all the drops (DA),
- the overall time average (TA),
- the combined speed average (SA).

Finally, the pupil calculates the speed based on DA and TA. Then, the results are compared. In most cases the results are different, depending on whether the pupil rounds the individual data, the intermediate results, or the final result.

- The teacher might point out to the pupil that the inaccuracies that result from rounding the numbers are cumulative throughout the exercise.

1.1.3 To EXPRESS a given decimal number in scientific notation and vice-versa.	III	- With calculators and computers, scientific notation is often used in order to provide an adequate approximation of a given number. It is therefore essential that the pupil learn to transform a decimal number into a number expressed in scientific notation, and vice-versa. He should also be able to recognize the range of accuracy of scientific notation.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.3 To EXPRESS a given decimal number in scientific notation and vice-versa (cont'd).	III	- Until recently, the number 13 475 432,4 expressed in scientific notation was written as follows: 1,347 543 24 x 10^7. On calculators and computers, the symbol 1,347 543 24 E 7 is used to designate the same number (here E 7 means x 10^7). Because of the increasing use of calculators and computers, the preferred way of writing the number 13 475 432,4 in scientific notation is 1,347 543 24 E 7. - In scientific notation the number n is written as m E e, where $\|m\| \in [1, 10[$ and $e \in \mathbb{Z}$. The theoretical relation between the numbers n, m, and e can be expressed as $n = m \text{ E } e$. Thus, 1439,72 = 1,439 72 E 3, 0,098 = 9,8 E -2, -329,4 = -3,294 E -2. - It is important to draw the pupil's attention to the fact that scientific notation makes it possible to achieve a certain degree of accuracy (generally from 8 to 10 significant figures), and that for numbers with more than eight or ten significant figures, a certain degree of imprecision must be tolerated. Consider these examples: (a) 123 x 456 = 5,6088 E 4 (b) 17,3 x 0,021 = 3,633 E -1 In these two problems, the answers expressed in scientific notation correspond exactly to the results of the operations. However, 123 456 789 x 123,456 789 $\approx$ 1, 524 157 875 E 10, and the symbol $\approx$ is read "... is approximately equal to ..." There can be no equality between 123 456 789 x 123 456 789 and 1,524 157 875 E 10 because the product of the two numbers above contains 17 significant figures: For the same reasons, $\frac{2}{3} \approx$ 6,666 666 667 E -1 because $\frac{2}{3} = 0,\overline{6}$. - The pupil can use the calculator to check whether the translation from one notation to another is correct.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.3 To EXPRESS a given decimal number in scientific notation and vice-versa (cont'd).	III	- In the following example, scientific notation is used to describe certain characteristics of the earth: . The great axis of the earth's orbit is approximately 2,99 E 8 km. . The diameter of the earth at the equator is approximately 1,2756 E 4 km. . The diameter of the earth at the poles is approximately 1,2713 E 4 km. . One second is approximately equivalent to 3,1688 E -8 year. . etc. - In BASIC, the instruction PRINTUSING "##, ##!!!!"; is used for writing the number N in scientific notation by rounding the second decimal. If N = 2357, this number will be printed as 2,34 E + 03.
1.1.4 To ROUND the result of an operation performed on decimal numbers to a given degree of accuracy.	III	- In the first cycle of secondary school, the pupil has learned to round numbers: - to the nearest hundred; - to the nearest ten; - to the nearest unit; - to the nearest tenth; - to the nearest hundredth. - In the second cycle of secondary school, the pupil should be able to round numbers to any given degree of accuracy. The ability to round numbers is essential for completing Objective 1.1.5. Example: Find the purchase price of a car, rounded to the nearest hundred, if the list price is $8795 minus a 5% discount. Solution: 8795 - (8795 x 0,05) = 8355,25 Answer: The purchase price is approximately $8400. - It is possible to make a quick mental estimation of the result if rounding techniques are used for each number in the expression to be evaluated and if the operation is performed on these rounded values. Thus, the expression 8795 - (8795 x 0,05) could become 8800 - (9000 x $\frac{5}{100}$), and the result is 8800 - 450 = 8350. This estimation technique is used in Objective 1.1.5.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.4 To ROUND the result of an operation performed on decimal numbers to a given degree of accuracy (cont'd).	III	- As a general rule, when the calculator is used to operate on decimal numbers, the machine records the results of the operations before displaying them. For example, $\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ is associated with 0,999 999 9 and not with 1, because the calculator truncates results, keeping only the first seven decimals and losing the others. This is an example of rounding to 0,000 000 1 by default. If the calculator displays 1 it is because it has rounded the result. - The different programming languages generally have instructions for rounding numbers. In BASIC, for example, 5,726 is rounded to 5,77 by means of the instruction PRINTUSING "##, ##"; N. - In order to ensure that the pupil understands the principle behind the rounding algorithm that is used, the teacher might ask him to write a program for rounding a number to a given degree of accuracy. In the following BASIC program such an algorithm is used. 10 CLS 20 INPUT "Indicates the number to be rounded"; N 30 INPUT "Rounding to what degree of accuracy"; A 40 N=N*A 50 IF N-INT(N)≥ 0.5 THEN N=INT(N)+1 ELSE N=INT(N) 60 N=N*A 70 PRINT 80 PRINTUSING "##########, ##########"; N 90 END - The following table is a simulated execution of this program for certain values of N and A.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.4 To ROUND the result of an operation performed on decimal numbers to a given degree of accuracy (cont'd).	III	

Values		Instructions		
N	A	40	50	60 and 80
537	10	53,7	54	540
532,6	100	5,326	5	500
0,123	0,01	12,3	12	0,12
0,178	0,11	1,78	2	0,2
-12,719	1	-12,719	-13	-13

- On line

40, the division is performed;
50, the result of the division is rounded to the nearest unit;
60, the result obtained on line 50 is multiplied;
80, the answer is printed.

- ENRICHMENT: Calculate the order of magnitude of the error due to truncation.

Compute $\frac{4}{11} + \frac{6}{9} + \frac{5}{7}$ if the numbers are truncated after

0,363 + 0,666 + 0,714 = 1,743.

A more accurate value is determined through this procedure:

4 [×] 9 [×] 7 [+] 6 [×] 11 [>] 7 [+] 5 [×] 11 [×] 9 [=] [÷] 11 [÷] 9 [÷] 7 [=]

The result is 1,744 588 7 (the exact value is 1,744 588).

The order of magnitude of the error is: 1,745 - 1,743 = 0,002, where 1,745 is the rounded value of 1,744 588 to the nearest thousandth.

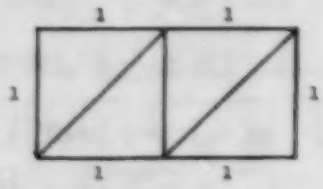
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.5 To ESTIMATE to a given degree of accuracy a product, a quotient, a square or square root, a cube or cube root of real numbers.	III	- This objective is intended to teach the pupil to look critically at the results obtained when using the calculator. Every time he performs a sequence of operations using an electronic instrument, he should be able to indicate the degree of accuracy of the result sought. - A good estimation of 123 x 459 would be 100 x 500 = 50 000, when the numbers are rounded to the nearest hundred. Actually, 123 x 459 = 56 457. - The pupil is taught how to estimate the square, square root, cube, and cube root of a real number because such a knowledge is useful for studying areas and volumes in geometry. - For verification purposes, the calculator can be used to extract the square root or the cube root of a given number. (a) The following is a sequence for extracting the square root of 25: 25 [log] [÷] 2 [=] [10^x] (b) This is a sequence for extracting the cube root of 27: 27 [log] [÷] 3 [=] [10^x] (c) On a computer, the expression EXP(1/3*LN 27) can be used for evaluating the cube root of 27. - Common logarithms (log) and natural logarithms (ln) are studied in the option courses. It is not necessary, therefore, to explain the meaning of the operations inherent in these procedures because the pupil does not have the necessary prerequisites. However, the sequences described in (a) and (b) can be explained to him so that he can use them for extracting square roots and cube roots. By generalizing, we obtain the following procedure: x [log] [x] n [=] [10^x] or x [ln] [x] n [=] [e^x]

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.5 To ESTIMATE to a given degree of accuracy a product, a quotient, a square or square root, a cube or cube root of real numbers (cont'd).	III	- Similarly, it is possible to find the *n*th power of x (x^n) by using the following procedure: x [log] [x] n [=] [10^x] or x [ln] [x] n [=] [e^x] - It is possible to present a wide variety of situations. This is an example: Estimate the number of seeds on the 64th square of a chessboard if 1 seed is placed on the first square, 2 on the second square, 4 on the third square, 8 on the fourth, and so on. If we observe the board, we discover the following pattern: Square Numbers: 1 2 3 4 ... 64 Seeds: 1 2 4 8 ... ... Seeds: 2^0 2^1 2^2 2^3 ... 2^{63} If we know that 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 = 1024 and is noted as 2^{10}, we can say that a good approximation of 1024 is 1000. Therefore, $2^{63} = 2^{10} \times 2^{10} \times 2^{10} \times 2^{10} \times 2^{10} \times 2^{10} \times 2^3$ $\approx 10^3 \times 10^3 \times 10^3 \times 10^3 \times 10^3 \times 10^3 \times 8$ $\approx 8 \times 10^{18}$ ≈ 8 E 18 On the calculator we have 9,2234 E 18.

INTERMEDIATE OBJECTIVES	LEVEL	NOTES
1.1.5 To ESTIMATE to a given degree of accuracy a product, a quotient, a square or square root, a cube or cube root of real numbers (cont'd).	III	- To estimate is to predict, with a varying degree of accuracy, such things as the value of a particular result, the magnitude of a number, the price of a house, the energy resources of a country, a person's age, the perimeter of a circle, the time, the lifetime of a star, the possibility of finding a job, the results of a coming election, etc. - To estimate also means to reach an opinion based on one's knowledge of the object to be estimated, or based on the experience one has acquired through one's contact with that particular object. A house painter estimates at a glance the number of litres of paint needed for painting a room. A cook estimates the amount of salt he/she should put in the stew. - We can estimate the height of a house by counting the number of stories if we know that each story is about 2,70 m tall. We can estimate that 2^{30} is approximately equivalent to one billion because we know that 2^{10} is approximately equivalent to one-thousand. We can estimate that the root of 120 is around 10, because we know that the root of 100 is 10. Others might say that the root of 120 is close to 11. - Generally, on the basis of an initial estimation and with the aid of additional information, we can make new estimations that will take us closer to the real value we wish to find. - To make an estimation or approximation, we use any method that makes it possible to come increasingly closer to the real value we are trying to find. - Here is an interesting activity. Choose a numerical expression. Ask the pupil to round each number so that he can do his calculations mentally. Then ask him to use the calculator to perform the same operations. Example: (132 x 48) + 4950 Round: (130 x 50) + 5000 Calculate mentally: 6500 + 5000 = 11 500 Calculate exactly: 11 286 It is possible to evaluate one's ability to estimate by calculating: $\frac{\text{estimated value} - \text{exact value}}{\text{exact value}} = \frac{\text{estimated value}}{\text{exact value}} - 1$

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.5 To ESTIMATE to a given degree of accuracy a product, a quotient, a square or square root, a cube or cube root of real numbers (cont'd).	III	By applying this formula to the above example, we find 0,019 by means of this sequence: (11500 − 11286) ÷ 11286 = or the simpler sequence: 11500 ÷ 11286 = − 1 = Since the value that is found is positive, this means that the estimation was greater than the exact value. The closer the result is to 0, the more accurate is the estimation. - If the pupil uses a calculator, he should acquire the habit of estimating the result that is sought. Estimating is a way of finding a plausible result. Sometimes it is enough to give the order of magnitude of the result. The more clues are used in the estimation method, the closer the estimated result is to the exact result. When we do an estimation before computing on the calculator we are predicting the result; when we do an estimation after using the machine, we are checking the plausibility of the result. - It is possible to estimate the result of operations quickly by translating numbers into scientific notation and using rounding techniques at the same time. Examples: 12350 + 17874 $\approx$ 1,0 E 4 + 2,0 E 4 = 3,0 E 4 = 30000 175000 − 13470 $\approx$ 1,8 E 5 − 1,3 E 4 = 18 E 4 − 1,3 E 4 = 16,7 E 4 = 167000 $\frac{3722 \times 52514}{280412} \approx \frac{4\text{ E }3 \times 5\text{ E }4}{3\text{ E }5} = \frac{20\text{ E }7}{3\text{ E }5} = \frac{20}{3}\text{ E }2 = 6,\overline{6}\text{ E }2 = 666,\overline{6}$ This technique requires the ability to estimate orders of magnitude, especially when dealing with addition and subtraction operations. Thus, in the second example, 1,8 E 5 − 1,3 E 4 gives 1,67 E 5, not 0,5 E 1.

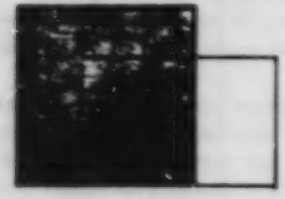
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.5 To ESTIMATE to a given degree of accuracy a product, a quotient, a square or square root, a cube or cube root of real numbers (cont'd).	III	- ENRICHMENT: - Estimate to a given degree of accuracy the square root of 2. - Evaluating $\sqrt{2}$ geometrically is equivalent to measuring the length of a square with an area of 2. With four right isosceles triangles it is possible to construct such a square without using the Pythagorean theorem.



This technique consists of using geometric construction for finding $\sqrt{2}$. To determine $\sqrt{2}$ by means of a decimal approximation we can use a ruler. However, as we are limited to 0,1 mm, we have to enlarge the figure, which is a practical problem.

Consider the following modified approach.

In a rectangle with an area of 2, $\sqrt{2}$ is found between the measures in the sides of the rectangle.



- The square has an area of 2.
- $\sqrt{2}$ is framed by the base and the height of the rectangle.

A rectangle with an area of 2 can have a base of 2 and a height of 1:
$(b_1 = 2, h_1 = 1)$.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.5 To ESTIMATE to a given degree of accuracy a product, a quotient, a square or square root, a cube or cube root of real numbers (cont'd).	III	To obtain a rectangle that is more "square" than the first, we might choose the one whose base is the arithmetic mean between the base and the height of the first.

Thus, $b_2 = \frac{b_1 + h_1}{2}$ where $h^2 = \frac{2}{b^2}$

$$b_2 = \frac{2 + 1}{2} = 1,5$$

$$h_2 = \frac{2}{1,5} \approx 1,333\ 333\ 3$$

And we proceed thus in a systematic manner:

$$b_3 = \frac{b_2 + h_2}{2} \approx 1,416\ 666\ 7 \qquad h_3 = \frac{2}{b_3} \approx 1,411\ 764\ 7$$

Since $\sqrt{2}$ is between b_3 and h_3, we must conclude that the decimal expression of $\sqrt{2}$ starts with 1,41. We only have to find certain procedures that make it possible to calculate the dimensions of rectangles that become more and more "square".

The following table illustrates the convergence toward $\sqrt{2}$:

Rectangles	Bases	Heights	Significant Figures
1	2	1	-
2	1,5	1,333 333 3	1
3	1,416 666 7	1,411 764 7	1,41
4	1,414 215 7	1,414 211 4	1,414 21
5	1,414 213 6	1,414 213 6	1,414 213

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.5 To ESTIMATE to a given degree of accuracy a product, a quotient, a square or square root, a cube or cube root of real numbers (cont'd).	III	Before looking for procedures, the expression to be calculated has to be set in a certain order. Since the base determines the height, we have: $bh = 2$ and $h = \frac{2}{b}$ Since the base of the next rectangle is $\frac{b + h}{2}$, we only have to calculate, for each value of b that is displayed, $\frac{1}{2}(b + \frac{2}{b})$, or $(b + \frac{2}{b})\frac{1}{2}$, or $(\frac{2}{b} + b)\frac{1}{2}$ Based on the value of b, we then execute one of these procedures: [+] [1/x] [x] 2 [=] [+] 2 [=] or [1/x] [+] [1/x] [+] 2 [=] (using calculators with algebraic logic that process operations in the order they are entered); or [STO] [1/x] [x] 2 [+] [RCL] [=] [+] 2 [=] or [STO] [(] [RCL] [+] [(] 2 [÷] [RCL] [)] [)] [÷] 2 [=]
1.1.6 Given two equal ratios, to ESTABLISH other proportions by applying the principal properties of equality of ratios.	IV	This is a dynamic way of handling a number such as $\sqrt{2}$. In addition, this way of proceeding has the advantage of being supported by a geometric interpretation at each step. This procedure is based on repetition. - In Secondary 2, the pupil was taught: "To DETERMINE ratios equivalent to a given ratio;" "To DETERMINE whether two given ratios form a true proportion;" "To FIND the value of a missing term in a proportion;" "To SOLVE a problem involving a simple proportion, given a realistic context."

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.6 Given two equal ratios, to ESTABLISH other proportions by applying the principal properties of equality of ratios (cont'd).	IV	- The ratios $\frac{a}{b}$ and $\frac{c}{d}$ form a proportion if and only if they are equal (not necessarily identical). If $\frac{a}{b}$ and $\frac{c}{d}$ are two ratios ($b \neq 0$ and $d \neq 0$), they form a proportion if and only if $ad = bc$, and we write $\frac{a}{b} = \frac{c}{d}$. - In a proportion $\frac{a}{b} = \frac{c}{d}$, • the terms a and d are called "extremes;" • the terms b and c are called "means." - Let $\frac{a}{b}$ and $\frac{c}{d}$ be equal ratios. Then, (a) reversing the two ratios gives a new proportion: $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{b}{a} = \frac{d}{c}$; $a \neq 0$, $b \neq 0$, $c \neq 0$, $d \neq 0$ (b) interchanging the extremes gives a new proportion: $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{d}{b} = \frac{c}{a}$; $a \neq 0$, $b \neq 0$, $d \neq 0$ (c) interchanging the means gives a new proportion: $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{c} = \frac{b}{d}$; $b \neq 0$, $c \neq 0$, $d \neq 0$ (d) by multiplying the two terms of a ratio by a non-zero quantity we can establish a proportion made out of the original ratio and the new ratio: $\frac{a}{b} = \frac{na}{nb}$; $n \neq 0$, $b \neq 0$

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.1.6 Given two equal ratios, to ESTABLISH other proportions by applying the principal properties of equality of ratios (cont'd).	IV	Example: What is the ratio of the circumferences of two circles with respective radii of 2,2 cm and 7 cm? $\frac{2\pi \times 2,2}{2\pi \times 7} = \frac{2,2}{7} = \frac{2,2 \times 5}{7 \times 5} = \frac{11}{35}$ (e) In a proportion, the sum of the numerators is to the sum of the denominators as a numerator is to its denominator: $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{a + c + e}{b + d + f}$; $b \neq 0$, $d \neq 0$, $f \neq 0$ Example: Three people invest respectively \$2500, \$3000, and \$4250 in a business partnership. If the first year's profits amount to \$5645 and are shared proportionately among the three investors, how much will each receive? $\frac{5645}{9750} = \frac{x_1}{2500}$ $\frac{5645}{9750} = \frac{x_2}{3000}$ $\frac{5645}{9750} = \frac{x_3}{4250}$ The profits will be distributed as follows: • The first partner will receive \$1447.44 • The second partner will receive \$1736.92 • The third partner will receive \$2460.64

TERMINAL OBJECTIVE 1.2 TO SOLVE real-life problems involving an algebraic equation (or inequation) of the first degree in one real variable.

In the first cycle of secondary school, the pupil learned "TO SOLVE problems involving common formulas or simple equations or inequations in one variable." (Objective 3.2.) The propositional forms that the pupils dealt with were simple and never involved situations requiring:

- the product of a positive or negative number by a negative number,
- the quotient of a positive or negative number by a negative number.

Such situations are found very rarely in everyday life. However, in Secondary 2 the pupil should be taught the rules of signs in a more structured way and he should be able to apply these rules to the solution of equations and inequations of the first degree in one variable.

It should also be pointed out that the reference set of the propositional forms presented to the pupil should be the set of real numbers, and that the coefficients or constants used in these propositional forms should also be real numbers.

Although this objective covers the same basic mathematical concepts (application of the properties of numbers, stability rules, etc.) as does Objective 3.2 in the Secondary 2 course, the reference set introduces a new dimension and new difficulties. Consequently, the rules of signs are taught more systematically at this point not for their own sake, but to enable the pupil to apply them for transforming one algebraic expression into another equivalent one.

Within the framework of the objectives covered in this section, various problems dealing with buying, selling, or percentages, in a business, consumer, or recreation context, can be solved. These problems can often be translated by an algebraic expression of the first degree in one variable.

Example: Lydia wants to buy a car that sells for $4750.00. How much will she spend if the sales tax is 8%?

The total cost of the car can be designated by X or any other letter.

Therefore, X = (4750,00 x 8%) + 4750,00 or

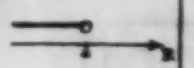
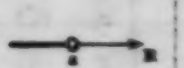
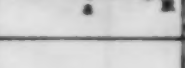
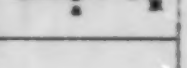
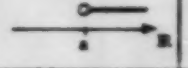
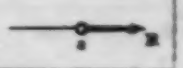
X = 4750,00 x 108%.

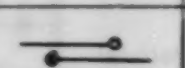
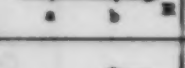
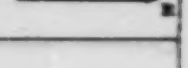
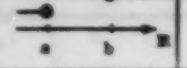
In this case X is an unknown rather than a variable since only one value can be associated with it.

An algebraic expression of the first degree in one real variable drawn from a problem is either an equation, an inequation, or a formula. In the first cycle the pupil has dealt with problems involving equations. The knowledge he has acquired in handling equations should be very useful for solving inequations. Such rules as:

- subtracting or adding the same quantity to each member, and
- multiplying or dividing each member by the same non-zero positive number,

are still valid for solving inequations. However, there is one difference at the level of the order axioms. Indeed, it is necessary to change the sense of the inequation when each member is multiplied or divided by a non-zero negative number: if $-2 < 8$, then $-2(-2) > -2(8)$, meaning $4 > -16$. There is another difference at the level of the solution set. For an equation of the first degree with one unknown, the solution set is a singleton, whereas for a simple or composite propositional form it is possible to find such situations as the following:

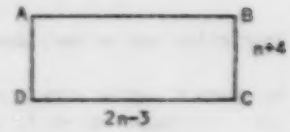
Propositional Forms	Intermediate Situations	Solutions
$x < a$		
$x \leq a$		
$x > a$		
$x \geq a$		

Propositional Forms	Intermediate Situations	Solutions
$x \geq a$ et $x < b$		
$x \geq a$ or $x < b$		
$x \leq a$ and $x > b$		
$x \leq a$ or $x > b$		

Example: Find the values n that will cause the perimeter of the rectangle ABCD to be less than 26 metres.

By applying the formula for determining the perimeter of a rectangle, it is possible to obtain this inequation:

$$2 [(2n-3) + (n+4)] < 26$$



If we analyze the situation carefully, we realize that "a side cannot have a zero or negative length." This imposes the following two additional constraints:

$$2n - 3 > 0 \quad \text{and} \quad n + 4 > 0.$$

The mathematical model which describes this situation therefore involves a system containing three inequations:

$$\left.\begin{array}{r} 2\,[\,(2n-3)+(n+4)\,] < 26 \\ 2n-3 > 0 \\ n+4 > 0 \end{array}\right\} \quad \text{equivalent to} \quad \left\{\begin{array}{l} n < 4 \\ n > 1{,}5 \\ n > -4 \end{array}\right.$$

This model can be represented as follows:

or:

Therefore: $1{,}5 < n < 4$,

and we have to conclude that n is to be found between 1,5 and 4 if we are to meet the conditions stated in the problem.

Problem-solving heuristics can be very useful to pupils for mathematizing a broad variety of situations. The teacher should make every possible effort to present pupils with a variety of situations that are taken from their environment, as in the following example:

A pupil obtains 75% of the 900 points granted for a set of examinations. What mark out of 100 should he obtain for his next examination if he is to have a cumulative average of 77%?

This is a classic type of problem. The solution technique presented below is less so, however:

Marks Added	Final Marks	Final Marks in Percentage
15	$900 \times 0,75 = 675$ $+ \frac{15}{690}$	$\frac{690}{1000} = 0,69 = 69\%$
		$675 + 15 = 690$
45	$900 \times 0,75 = 675$ $+ \frac{45}{720}$	$\frac{720}{1000} = 0,72 = 72\%$
		$675 + 45 = 720$
65	$900 \times 0,75 = 675$ $+ \frac{65}{740}$	$\frac{740}{1000} = 0,74 = 74\%$
		$675 + 65 = 740$
X	$900 \times 0,75 = 675$ $+ \frac{X}{675+X}$	$\frac{675+X}{1000} = 0,77 = 77\%$
		$675 + X = 770$

By trial and error, the pupil learns to mathematize a situation, that is to say, to translate it by an equation. In the past, the method used for solving this type of problem was intended to teach the pupil how to set up an equation. An equation such as $675 + X = 770$ can be solved by framing (tables), by reduction, or by repetition (see system solving).

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.2.1 To FIND the simplest equation (or inequation) corresponding to a given algebraic equation (or inequation).	III	- This objective is identical to Objective 3.1.7 in the first cycle program. It was dealt with in the Secondary II course. It is designed to enable the pupil to review the simple polynomials and to consolidate his knowledge and skills in this area. The pupil should be asked to perform addition, subtraction, multiplication, and division operations on simple polynomials. Example 1: $a + 7b + 2a + 3b$ Commutative property of addition $a + 2a + 7b + 3b$ $(1 + 2)\,a + (7 + 3)\,b$ Distributive property of multiplication over addition $3a + 10b$ Closure of addition It should be noted that justifications are not required. However, the pupil should have a good understanding of the properties involved. This process by which an expression undergoes a series of transformations is often called "reduction of similar terms," which is a misnomer. A better term would be "reduction of the expression." In some cases, as in the preceding example, a knowledge of factoring techniques may be necessary. The teacher should introduce the pupil to such techniques in order to give him the tools necessary for finding simpler equivalent expressions. It should be remembered that factoring involves the application of the distributive property: $a(b + c) = ab + ac$ and $ab + ac = a(b + c)$ because equality is a symmetrical relation. Example 2: $2(x + 4) - 26 = x + 12$ Distributivity of multiplication over addition $2x + 8 - 26 = x + 12$ $2x - 18 = x + 12$ Closure of addition $-x + 2x - 18 = -x + x + 12$ Additive inverse

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.2.1 To FIND the simplest equation (or inequation) corresponding to a given algebraic equation (or inequation) (cont'd).	III	$x - 18 = 0 + 12$ Closure of addition $x - 18 = 12$ Cancellation of addition $x - 18 + 18 = 12 + 18$ Additive inverse $x + 0 = 30$ Closure of addition $x = 30$ Cancellation law of addition This last equation is undoubtedly the simplest. Example 3: $-3x + 7 \leq 13$ Additive inverse $-3x + 7 - 7 \leq 13 - 7$ Closure of addition $-3x + 0 \leq 6$ Cancellation of addition $-3x \leq 6$ Simplifying $\frac{-1}{3} \times -3x \geq \frac{-1}{3} \times 6$ Closure of addition $1x \geq -2$ Cancellation of multiplication $x \geq -2$ In the last two examples a precondition for finding the simplest expression is a knowledge of the rules of inverse elements and of simplifying, and a knowledge of the order axioms, the properties of addition and multiplication operations (commutativity, associativity, cancellation property, symmetry, distributivity, and closure). - To enable the pupil to understand the role of x better, the teacher might ask him to evaluate the propositional form $3x + 5 < -9x - 8$ for several different values of x, and to indicate whether the resulting proposition is true or false.

x	$3x + 5$	<	$-9x - 8$	Propositions
-3	-4	<	19	true
-2,3	-1,9	<	12,7	true
-1,5	0,5	<	5,5	true
2	11	<	-26	false
18	59	<	-170	false

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.2.1 To FIND the simplest equation (or inequation) corresponding to a given algebraic equation (or inequation) (cont'd).	III	The following procedures can be used for evaluating each of the numbers of the equation: [x] 3 [=] [+] 5 [=] and [x] 9 [+/-] [=] [-] 8 [=] The command [=] is intended to separate the partial procedures. This exercise should enable the pupil to develop a better understanding of the concept of variable. - Let us assume the following propositional forms to be equivalent: $3x + 5 < -9x - 8$ $3x + 9x + 5 < -9x + 9x - 8$ $12x + 5 < 0 - 8$ $12x + 5 < -8$ $12x + 5 - 5 < -8 - 5$ $12x + 0 < -13$ $12x < -13$ $x < \frac{-13}{12}$ $x < -1,08\overline{3}$ The pupil can be asked to verify whether for a given value of x all these propositional forms are true or false. They are all true if x is replaced by a number lower than $-1,08\overline{3}$. It is important to point out to the pupil that two propositional forms can be true for a given value of x without being equivalent. The propositional forms $3x + 5 < 17$ and $3x < 17 + 5$ are true for $x = 3$, but they are not equivalent. - Generally, when looking for equivalent expressions to a given expression, the calculator can be used for verification purposes, as in the following example: The expression $a + 7a - b + 2b - a - 3b + a - b$ is simplified to $8a - 3b$.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.2.1 To FIND the simplest equation (or inequation) corresponding to a given algebraic equation (or inequation) (cont'd).	III	If the two expressions are equivalent, we should obtain the same result if we replace a and b by any two values. If the answers are different, each intermediate step should be checked in order to find out where an error has occurred. - By using the calculator, the pupil will be encouraged to apply this verification procedure, especially when dealing with realistic situations. - It is important to focus only on the simple polynomials. The teacher should keep in mind, especially with respect to multiplication and division operations on polynomials, that the pupil has not had much experience working with exponents. - In Secondary 2, the pupil has probably learned these two laws: $\forall b \in \mathbb{Z}^*, \forall m, n \in \mathbb{Z} : b^m \cdot b^n = b^{m+n}$ (Objective 3.1.5) $\forall b \in \mathbb{Z}^*, \forall m, n \in \mathbb{Z} : \frac{b^m}{b^n} = b^{m-n}$ (Objective 3.1.6) He should at least have had the opportunity to apply the definition of power: $3a^2 \times 7a = 3aa \times 7a = 3 \times 7 \times aaa = 21a^3$ $\frac{18b^3}{6b^2} = \frac{18bbb}{6bb} = \frac{\overset{3}{\cancel{18}}\cancel{b}\cancel{b}\cancel{b}}{\underset{1}{\cancel{6}}\cancel{b}\cancel{b}} = 3b$ - It is important to introduce the pupil to the law expressed as $\forall b \in \mathbb{Z}^* : b^0 = 1$ using examples in which the second law and the definition of power are applied. Thus, $\frac{b^3}{b^3} = b^{3-3} = b^0$ Second law on exponents $\frac{b^3}{b^3} = \frac{bbb}{bbb} = 1$ Definition of power therefore, $b^0 = 1$ On the basis of the transitivity of equality

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.2.1 TO FIND the simplest equation (or inequation) corresponding to a given algebraic equation (or inequation) (cont'd).	III	- Polynomials and the theory of exponents will be studied in greater detail in the Option I Mathematics course. At this point it is advisable to cover only simple expressions.
1.2.2 To IDENTIFY the rule(s) used to isolate the unknown in an algebraic equation (or inequation) of the first degree in one variable.	III	- The following rules can be identified: . Balancing: - We can add or subtract a given quantity from each member of an equation or inequation. . Simplifying: - We can multiply or divide each member of an equation by a given quantity; - We can multiply or divide each member of an inequation by a given positive quantity; - We change the sequence when we multiply or divide each member of an inequation by a negative quantity. Examples: (a) $-3x + 7 \leq 13$ $-3x \leq 6$ $x \leq -2$ (b) $-3x + 7 \leq 13$ (1) $(-3x + 7) \leq 13$ (2) $(-3x + 7) - 7 \leq 13 - 7$ (3) $-3x + 7 - 7 \leq 13 - 7$ (4) $-3x + (7 - 7) \leq 13 - 7$ (5) $-3x + 0 \leq 6$ (6) $-3x \leq 6$ (7) $(-3x) \leq 6$ (8) $(-1/3) \times (-3x) \geq (-1/3) \times 6$ (9) $(-1/3 \times -3)x \geq (-1/3 \times 3) \times 2$ (10) $+1x \geq 1 \times 2$ (11) $+x \geq -2$ (12) In example (a) the pupil uses an abbreviated method, keeping only steps (1), (7), and (12) of the detailed solution presented in example (b).

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.2.2 To IDENTIFY the rule(s) used to isolate the unknown in an algebraic equation (or inequation) of the first degree in one variable (cont'd).	III	This method reminds us of the two following well-known rules: - when we change sides we change signs; - we change the sense of the inequation (of the order) when we multiply (or divide) each member by a non-zero negative number. The phrase "when we change sides we change signs" means that we perform the inverse operation.
1.2.3 To EXPRESS by expansion or reduction the solution set of a propositional form of the first degree in one variable.		- To find the solution set it is necessary to determine the reference set first. Thus, for a propositional form such as "x is the mayor of..." any reference set such as the set of the planets, the set of the months of the year, or any other set of this type, will produce a set of false propositions because no value of these sets is a solution of this propositional form. A reference set should be selected with care. For example, it would not be appropriate to select $\mathbb{Z}$ as the reference set for the equation $5x + 4 = 3x + 7$. - Generally, the pupil should work with the set of real numbers as the reference set. - The solution set of $5x + 4 = 3x + 7$ is $\left\{\frac{3}{2}\right\}$. The pupil should be required to provide a good answer. Thus $\frac{3}{2}$ would not be an acceptable answer because the pupil is asked to find the solution set and not the solution of the equation. - The calculator can be used for finding the solution set of an equation of the type $ax + b = c$. The simplest equivalent equation is $x = \frac{c - b}{a}$ and the solution set $\left\{\frac{c - b}{a}\right\}$. If we evaluate $\frac{c - b}{a}$ with the aid of the calculator it is possible to find the solution we are seeking quickly. - The Roman balance is another method for solving an equation by finding a satisfactory approximation of the value of the variable x. Consider the equation $5x - 4 = 2x + 6$. One tray of the balance holds $5x - 4$ and the other tray contains $2x + 6$. The behaviour of the balance when x is given a value helps us find another value for x. And little by little, we come closer to the particular value which brings about a certain equilibrium. For each value of x we calculate $5x - 4$ and $2x + 6$.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.2.3 To EXPRESS by expansion or reduction the solution set of a propositional form of the first degree in one variable (cont'd).	III	

x	5 x - 4	Balance	2x + 6
-1	-9		4
0	-4		6
1	1		8
2	6		10
3	11		12
4	16		14
5	21		16

Beginning with x = 3 the position of equilibrium of the balance changes. We construct a new table similar to the first, but with values between 3 and 4.

x	5 x - 4	Balance	2x + 6
3,0	11,0		12,0
3,1	11,5		12,2
3,2	12,0		12,4
3,3	12,5		12,6
3,4	13,0		12,8
3,5	13,5		13,0

Starting at 3,4 the position of equilibrium of the balance changes. We again construct a new table, using values between 3,30 and 3,40. We proceed in this manner until we are satisfied with the approximate answer, to the nearest thousandth, for example, or 3,333.

The pupil should be required to find various methods for checking his calculations. In this case, the composites of inverse procedures can be used quite effectively. Procedure B is the inverse of procedure A.

- This is how we might write a BASIC program for finding the solution of an equation of the first degree with one unknown:

```
10 CLS
20 INPUT "Enter the values associated with the coefficients of Ax+B=C"; A, B, C
30 IF C-B=0 THEN IF A=0 THEN PRINT "There are countless possible solutions." ELSE PRINT
   "There is no solution."
40 IF A=0 THEN STOP ELSE X=(C-B)/A
50 PRINT "The solution of this equation is "; X
60 END
```

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.2.4 To GRAPH a propositional form of the first degree in one variable.	III	- It is strongly recommended that the pupil be introduced to the concept of interval. For example,

$\forall a,\ x \in \mathbb{R} : x \geq a$ is represented by / is noted by $[a, +\infty[$

$\forall a,\ x \in \mathbb{R} : x > a$ is represented by / is noted by $]a, +\infty[$

$\forall a,\ x \in \mathbb{R} : x \leq a$ is represented by / is noted by $]-\infty, a]$

$\forall a,\ x \in \mathbb{R} : x < a$ is represented by / is noted by $]-\infty, a[$

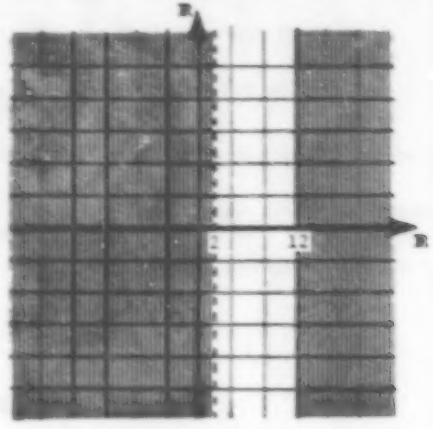
$\forall a, b,\ x \in \mathbb{R} : a \leq x \leq b$ is represented by / is noted by $[a, b]$

$\forall a, b,\ x \in \mathbb{R} : a < x \leq b$ is represented by / is noted by $]a, b]$

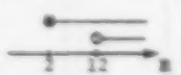
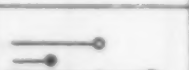
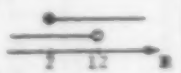
$\forall a, b,\ x \in \mathbb{R} : a \leq x < b$ is represented by / is noted by $[a, b[$

$\forall a, b,\ x \in \mathbb{R} : a < x < b$ is represented by / is noted by $]a, b[$

- The propositional form "$x < 2$ and $x \geq 12$" is represented on a line as:

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.2.4 To GRAPH a propositional form of the first degree in one variable (cont'd).	III	- In addition to this type of representation, the Cartesian plane can be used advantageously for representing a propositional form of the first degree in one variable. For our purposes, we can consider the propositional forms " $X<2$ and $X \geq 12$" and "$X + 0y < 2$ and $X + 0y \geq 12$" equivalent. The following graph shows how these equivalent propositional forms can be represented on the coordinate plane:
		
		- The two graphic representations show clearly that the solution set is empty. The pupil should be made aware that the preceding representations are those of the propositional form and not of its solution set.
		- By extrapolating it, is possible to construct a space graph, $X<2$ being associated with $X + 0y + 0z < 2$, and $X \geq 12$ being associated with $X + 0y + 0z \geq 12$.
1.2.5 To GRAPH the solution set of a propositional form of the first degree in one variable.		- Let $X < 2$ and $X \geq 12$ be a propositional form represented by the following graph:

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.2.5 To GRAPH the solution set of a propositional form of the first degree in one variable (cont'd).	III	- If we examine the preceding graph we realize that the solution set is empty, because there is no value that meets the condition. This solution set is represented by the following graph: We indicate that the solution set is empty by means of the symbol $\emptyset$ or $\{\}$. The following table shows a number of situations:

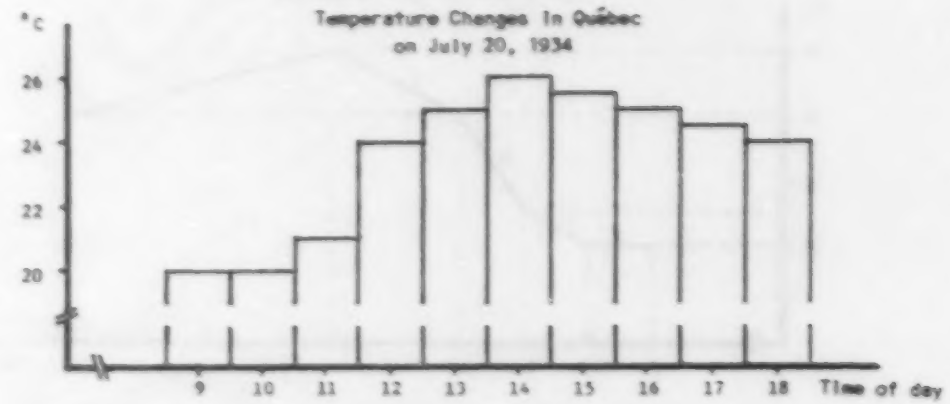
Propositional Forms	Intermediate Situations	Solutions
$x > 2$		
$x \geq 2$ and $x > 12$		
$x \leq 2$ and $x < 12$		
$x \geq 2$ and $x < 12$		

- It is important that the pupil be able to distinguish between the representation of a propositional form and that of its solution set.

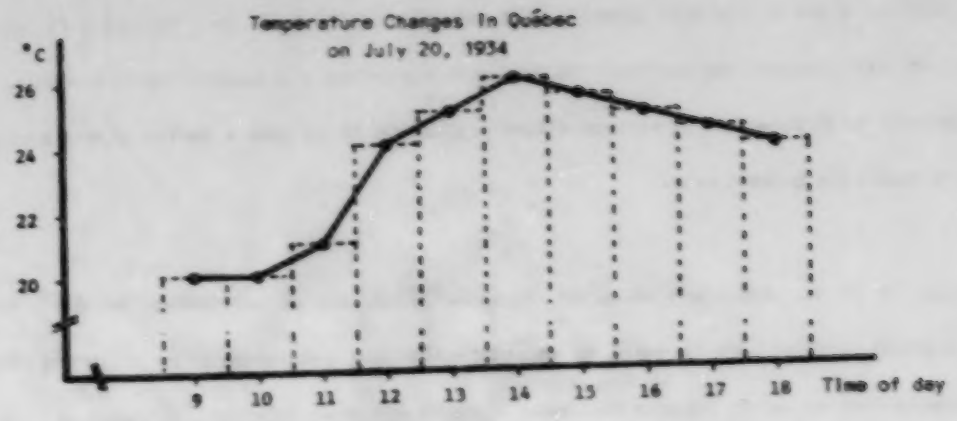
TERMINAL OBJECTIVE 1.3 TO SOLVE problems of everyday life involving a constant or affine function.

The concept of function is one of the most commonly used mathematical concepts in a contemporary world that is increasingly influenced by a steady flow of technological innovations. Objective 1.3 provides a framework for introducing the pupil to this concept and for developing his ability to discover it in various situations, using it to gain a better understanding of these situations, to compare them, and to solve them with greater ease.

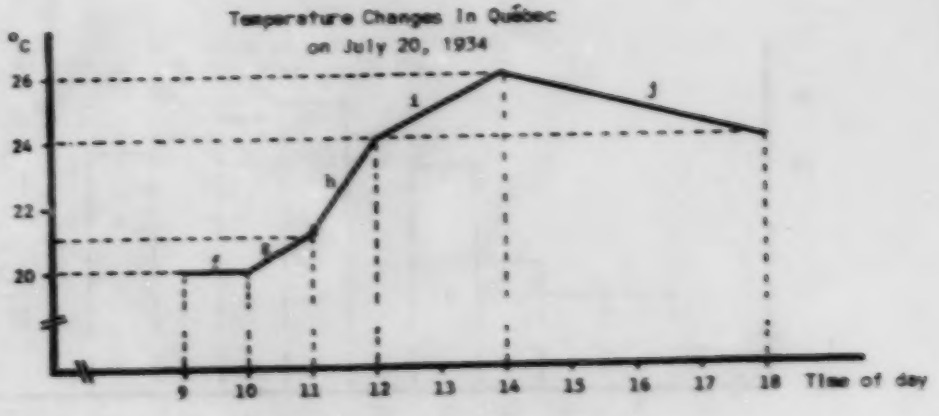
To study functions, it is not absolutely necessary to have a knowledge of set theory, of Cartesian products, and of the concept of relation. The approach adopted here is based on observation of the phenomenon under study by means of its graphic representation (Cartesian representation) so as to isolate its basic characteristics. Consider, for example, the following histogram representing temperature changes in Québec on July 20, 1934.



From this histogram it is possible to construct the following broken-line diagram:



It is then possible to define the functions f, g, h, i, and j by restricting the domains of the functions appropriately:



It becomes possible to evaluate the variation rate of each of these functions:

Functions	Domains	Variation Rates	
f	$9 \leq x \leq 10$	$\frac{20 - 20}{10 - 9} = \frac{0}{1} = 0$	0
g	$10 \leq x \leq 11$	$\frac{21 - 20}{11 - 10} = \frac{1}{1} = 1$	1
h	$11 \leq x \leq 12$	$\frac{24 - 21}{12 - 11} = \frac{3}{1} = 3$	3
i	$12 \leq x \leq 14$	$\frac{26 - 24}{14 - 12} = \frac{2}{2} = 1$	1
j	$14 \leq x \leq 18$	$\frac{24 - 26}{18 - 14} = \frac{-2}{4} = -\frac{1}{2}$	$-\frac{1}{2}$

It will be noted that a temperature rise is associated with a positive slope (increase, variation rate), and that inversely, a decrease in temperature is associated with a negative slope. In addition, the absolute value of the variation rate indicates the speed at which the temperature change occurred.

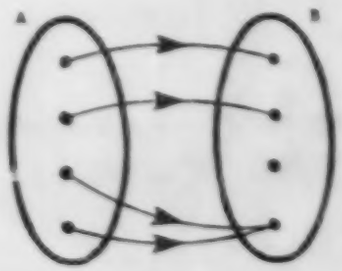
This terminal objective has many possible applications. It would be advantageous, therefore, to use a broad variety of situations taken from the pupil's everyday experience, or pertaining to the business world, the pure sciences, technology, the human sciences, or the applied sciences.

These functions must always be studied in conjunction with their graphic representations and in the context of the situations from which they arise. The preferred approach is to "mathematize a situation, operate on the model, and interpret the results." The teacher is advised to encourage the pupil to use different problem-solving heuristics.

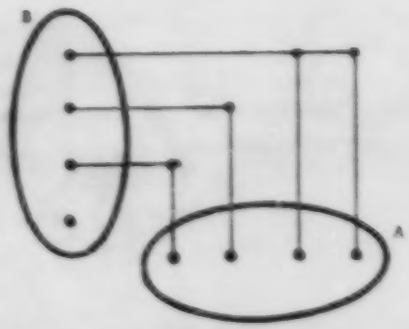
It is strongly recommended that the pupil use a pocket calculator, as this instrument makes it possible to construct graphs and to calculate variation rates more quickly. It is also suggested that the teacher be very strict with the pupil regarding precision in the construction of graphs.

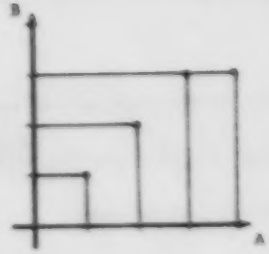
The supplementary teaching notes provide additional information on affine functions as well as a number of examples of applications that are likely to be helpful guideposts in the search for "useful functions."

In Bouvier's Dictionnaire des mathématiques, a function is described as being synonymous with application. This sense of the concept has been adopted in this instructional guide. It should be specified, for the teacher's benefit, that a function is a relation for which any object x of its domain is associated with a single object y of its range such as $(x, y) \in f$. Given two sets A and B, a function f will be defined from A to B if $\forall x \in A, \exists_1 y \in B: \forall (x, y) \in f$.



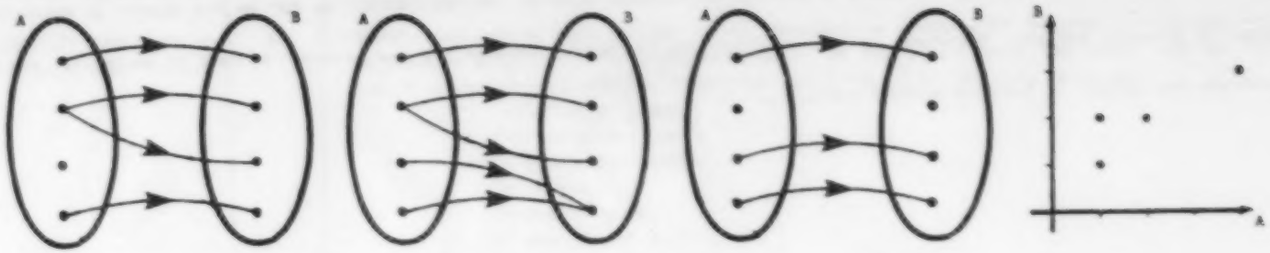
Arrow Graph



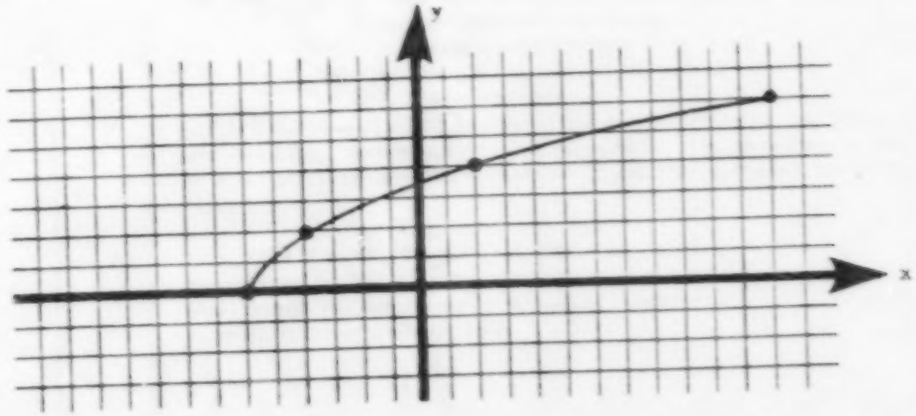


Cartesian Graph (Networks)

Consider the following situations:



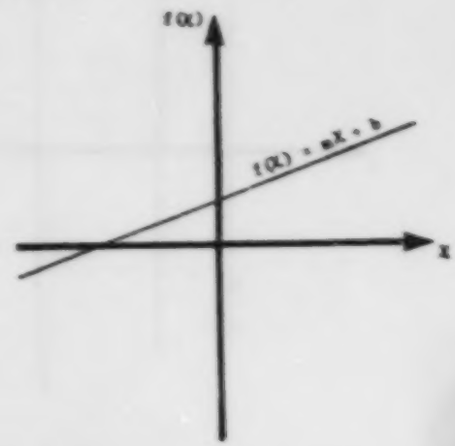
They do not represent functions but relations. However, the Cartesian graph that follows is generally considered to represent a function; its domain is given implicitly by the graph.



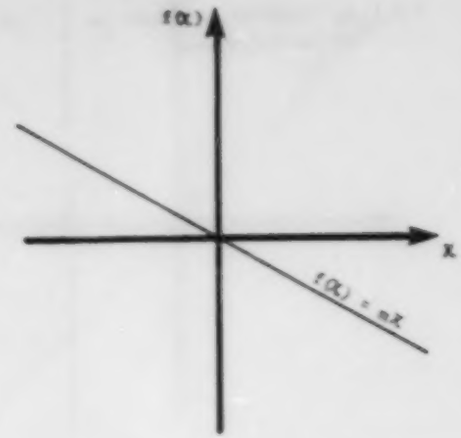
The domain of this square root function is $[-3, +\infty[$ and this function can be defined by means of the rule of correspondence:

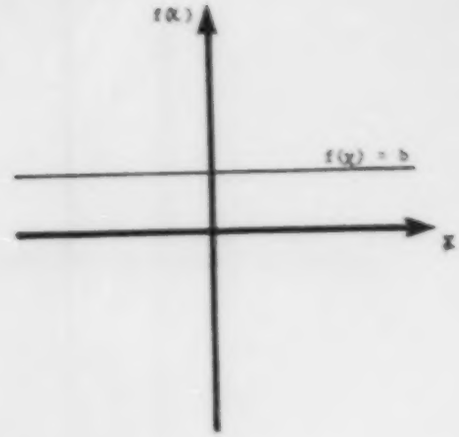
$$f : [-3, +\infty[\longrightarrow \mathbb{R} : x \longmapsto \sqrt{x + 3}$$

it would be incorrect to define this correspondence as being a function from $\mathbb{R}$ to $\mathbb{R}$. In secondary school, in terms of teaching there does not seem to be any advantage in making a distinction between function and application as far as the domain is concerned. The teacher is strongly advised, therefore, not to highlight the difference between the two in order to make it easier for the pupil to assimilate the concept of function without ambiguity.

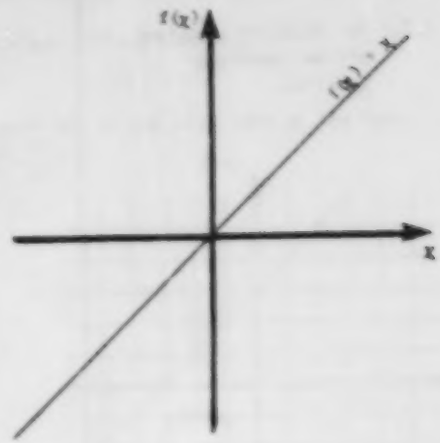
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.1 To IDENTIFY constant or affine functions.	IV	- Before introducing this subject it is advisable to review with the pupil the vocabulary associated with Cartesian representation. In the first cycle the pupil learned "TO LOCATE an object on a surface" (Terminal Objective 4.5). The main terms are: - Cartesian plane, - X-axis (abscissa), - y-axis (ordinate), - origin, - ordered pair, - abscissa of a pair, - ordinate of a pair, - coordinates of a pair. - A constant function or a first-degree function is represented in the Cartesian plane by a line that is parallel to the X-axis. - The general form of this type of function is given by: $f(X) = mX + b$, where X is the independent variable; f(X) is the dependent variable ($f(x) = y$); m is the slope or rate of variation; b is the y-intercept; and its representation is a line in the Cartesian plane. In $f(X) = mX + b$, X and f(X) are variables, whereas m and b are parameters. The parameters m and b determine a line in a set of lines (sheaf or direction); X and f(X) designate the points (X, f(X)) of this line. - If $m \neq 0$ and $b \neq 0$, the function $f(X) = mX + b$ is represented in the plane by a line that does not pass through the origin. This polynomial function is known as an affine function. 

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.1 To IDENTIFY constant or affine functions (cont'd).	IV	- If $m \neq 0$ and $b = 0$, the function $f(x) = mx + b$ becomes $f(x) = mx$ and it is represented graphically by a line that passes through the origin. This function is called a direct-variation case of the affine function.
		- If $m = 0$ and $b \neq 0$, the function $f(x) = mx + b$ becomes $f(x) = b$, and is represented graphically by a line that is parallel to the x-axis. This function is called a constant function. Its degree is zero.

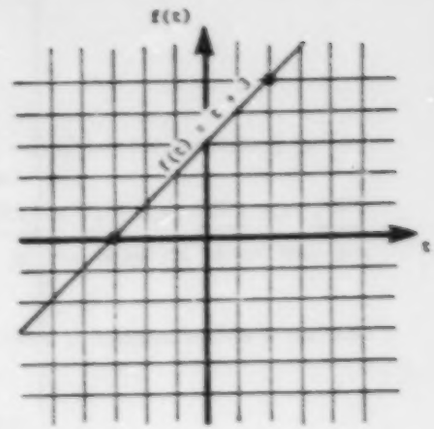




INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.1 To IDENTIFY constant or affine functions (cont'd).	IV	- If $m = 1$ and $b = 0$, the function $f(x) = mx + b$ becomes $f(x) = x$, and is represented graphically by a line that bisects the angle formed by the x- and y-axes in the first quadrant. This function is called an identity function. This linear function is a particular case of the affine function.
		- After a number of evaluations it should be pointed out to the pupil that a given value of "x" makes it possible to discover one and only one value of "f(x)". It is on the basis of this characteristic that $f(x) = mx + b$ is said to be a function.
		- In the Cartesian graph opposite, for a given value of "x_1" there is an infinite number of corresponding values for "y". The graph is therefore not the graph of a function.



INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.1 To IDENTIFY constant or affine functions (cont'd).	IV	- An affine function can be identified by means of: - its Cartesian graph, - its equation, - a contextualized problem. - Here is such a problem: The wheels of a racing bicycle have a diameter of 66 cm. At a certain speed, after applying the brakes to the two wheels the cyclist has to go a distance of 1,74 m before he can come to a full stop. What is the total distance he covers for a given number of turns of the wheels? This function is an affine function: f(t) = 66π t + 174 where: t indicates the number of wheel turns, f(t) gives the total distance expressed as a function of the number of wheel turns: 174 because 1,74 m = 174 cm. - A BASIC program using multiple unconditional branchings can be used for identifying the different types of functions. 10 CLS 20 INPUT "Enter the values of m and b in f(x)=mx+b"; M, B 30 IF M<>0 THEN GOTO 50 ELSE PRINT "This function is constant". 40 STOP 50 IF B=0 THEN PRINT "This is a direct variation or linear function". ELSE PRINT "This is an affine function". 60 END
1.3.2 To DETERMINE the image for a given value of the domain of a constant or affine function.	IV	- The domain of a function can be defined as the set of values which the independant variable may take.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.2 To DETERMINE the image for a given value of the domain of a constant or affine function (cont'd).	IV	- In the case of affine functions, the domain can be the set of real numbers. - It is important to point out to the pupil that the pair (x, y) can also take on the form (x, f(x)). - An affine function can be evaluated: . by means of its Cartesian graph: "Evaluate f(t) for t = 2." Answer: 5 This evaluation is often imprecise if the equation associated with the graph is not given.  . by means of its equation: "Evaluate f(t) = 2πrt + 174 for t = 250, π ≈ 3,14 and r = 33." Answer: f(250) = 51 984 because 2 x 3,14 x 33 x 250 + 174 = 51 984. . By means of a contextualized problem: The wheels of a racing bicycle have a diameter of 66 cm. At a certain speed, after applying the brakes to the two wheels the cyclist has to go a distance of 1,74 m before he comes to a full stop. What is the total distance he covers if the front wheel makes 400 turns? f(t) = 66πt + 174 f(400) ≈ 83 070 because 66 x 3,14 x 400 x 174 = 83 070. Answer: The bicycle rider covers a distance of 83 070 or 0,83 km. When a function is evaluated by means of a contextualized problem, it is important to require the pupil to interpret the numerical value that is obtained.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.2 To DETERMINE the image for a given value of the domain of a constant or affine function (cont'd).	IV	- Here is a contextualized problem for which the calculator is very useful. A container holds 58 cm^3 of oil that has a mass of 46,4 g. Complete the following correspondence table.

Oil volume in cm^3	1	2,5	17,5	34,5	58	79	83
Oil mass in g					46,4		

We can enter the linearity coefficient (the slope obtained by means of the quotient mass/volume: $\frac{46,4}{58} = 0,8$) into the calculator's memory by using the sequence 46,4 ÷ 58 [M].

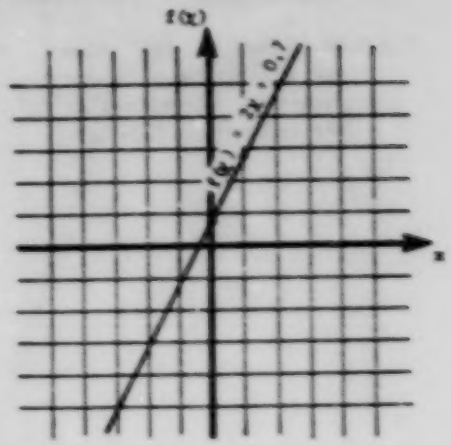
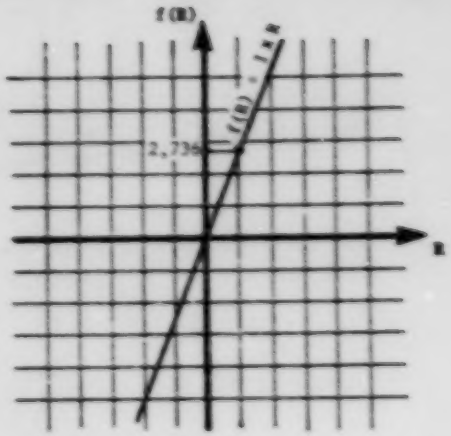
To complete the table we can then use the procedure [ ] [×] [MR] [=], or the procedure that does not utilize the memory: P: [ ] [×] 0,8 [=].

- It subsequently becomes possible to develop the concept of inverse function by using the concept of inverse procedure.

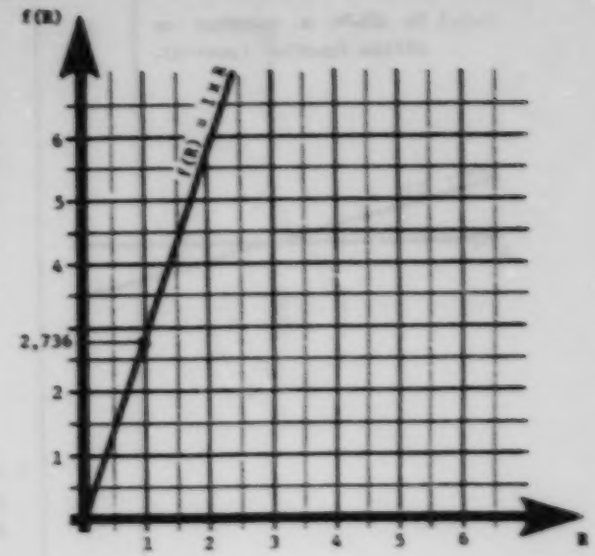
If f(a) = 14, the value of a is found by means of the procedure P^{-1}: [ ] [÷] 0,8 [=].
P^{-1} is found by means of the following transformations:
- to P: [ ] [×] 0,8, we associate the function f(X) = 0,8X;
- when we look for the inverse of this function we find:

$$y = 0,8x$$
$$x = 0,8y$$
$$y = \frac{x}{0,8}$$
$$y = \frac{5x}{4}$$
$$f^{-1}(x) = \frac{5x}{4}$$

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.2 To DETERMINE the image for a given value of the domain of a constant or affine function (cont'd).	IV	- which enables us to find the inverse procedure F^{-1}: [ ] [x] 5 [÷] 4 [=]. - Here is another example: $g: x \longmapsto 12{,}85 \div 14{,}17$ A: [ ] [x] 12,85 [=] [÷] 14,17 [=] is an example of procedure that can be used for evaluating the function g for any given value. There are other procedures such as: B: [ ] [STO] A [x] 12 [=] [÷] 14 [=] [÷] [RCL] A [x] 0,85 [+] 0,17 [=] (with a calculator that performs operations in the order in which they are entered). What is important is for the pupil to understand that these procedures evaluate the same function. Thus, $x \longmapsto 12{,}85x \div 14{,}17$ (translation of A) $x \longmapsto 12x \div 14 \div 0{,}85x \div 0{,}17$ (translation of B) are different expressions for describing the same function. The procedures associated with these expressions are only different means of determining images by this same function. - The teacher might ask the pupil to prepare a program for evaluating a function for a given value of its domain. The following is an example of such a program written in BASIC: 10 CLS 20 INPUT "Indicate the values of a and b in the function f(X)=aX +b."; A, B 30 INPUT "Indicate the value of the domain of this function"; X 40 Y=A*X+B 50 PRINT "f("; X ; ")="; Y 60 END

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.3 To GRAPH a constant or affine function.	IV	- A graph of a function can be constructed based on: . the equation of the function: "Graph the following function: $f(x) = 2x + 0,7$." $f(0) = 0 + 0,7$ $= 0,7$ we have (0, 0,7) $f(1) = 2 \times 1 + 0,7$ $= 2,7$ we have (1, 2,7)  . a contextualized problem: "The electric current I in a given electric circuit measure 2,736 amperes. Draw the curve representing the tension (in volts) as a function of the resistance R (in ohms). This function is defined by: $f(R) = I \times R$." 

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.3 To GRAPH a constant or affine function (cont'd).	IV	- It should be noted that in the graphs associated with most everyday-life situations or scientific problems, only the first quadrant of the Cartesian plane is used. In the above example it is obvious that resistance cannot have a negative value. Therefore f(R) is always positive, and this function can be represented by the graph shown here.



- Based on the preceding comment, it would be of interest to point out to the pupil that the mathematical model created for interpreting real situations is often more general (broader), and that one should focus only on the significant part of the model.

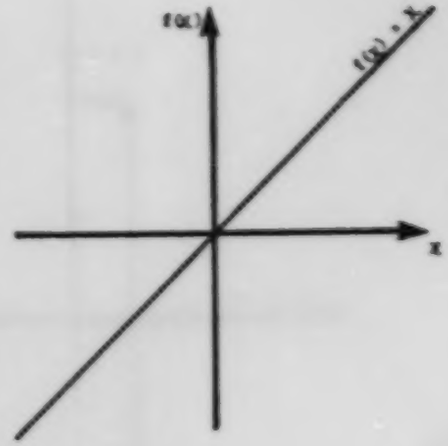
- Certain affine functions can be evaluated more easily with the help of a calculator and, consequently, they can be graphed more rapidly. The systematic use of the calculator should also make it possible to present more meaningful situations to the pupil.

- Point out to the pupil that if we let $X = 0$ in $f(X) = aX + b$, it becomes possible to determine the point where the line crosses the y-axis:

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.3 To GRAPH a constant or affine function (cont'd).	IV	$f(0) = a \times 0 + b$ $= 0 + b$ $= b$ which makes it possible to obtain the pair (0, b). On the equation line $f(x) = ax + b$, "b" is the ordinate y-intercept.

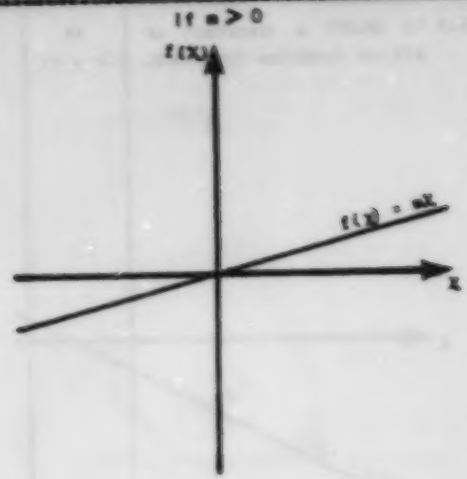
- In order to explore in greater detail certain characteristics of the affine functions presented earlier, it might be interesting to propose a sequence of activities to the pupil such as the following:

(a) Draw the Cartesian graph of the identity function.

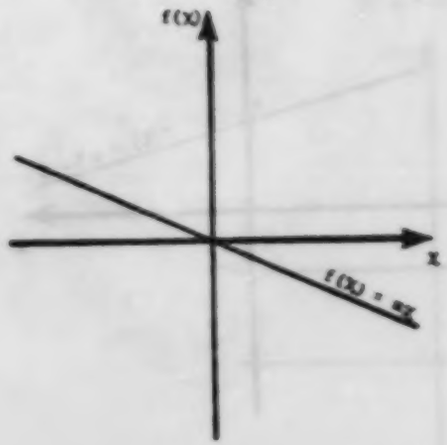


INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.3 To GRAPH a constant or affine function (cont'd).	IV	(b) Draw the Cartesian graph of the linear (direct variation) or constant function.

If $a > 0$



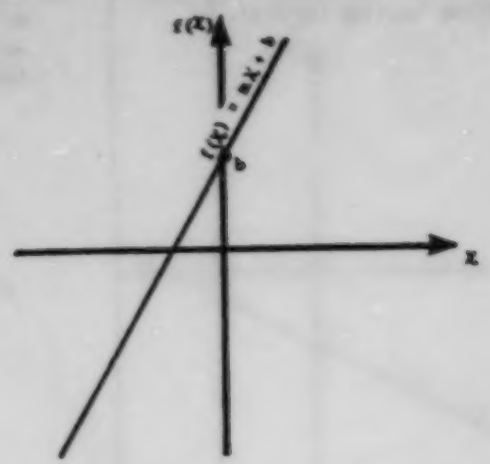
If $a < 0$



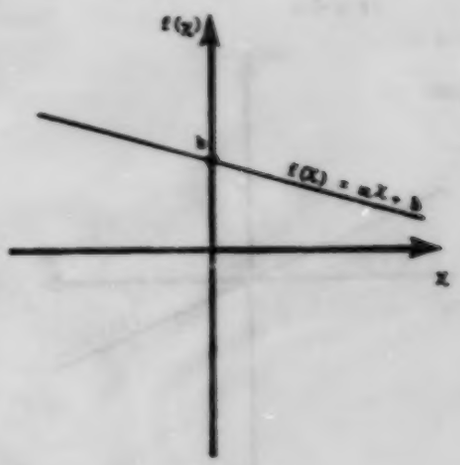
If $a = 0$

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.3 To GRAPH a constant or affine function (cont'd).	IV	(c) Draw the Cartesian graph of the affine or constant function.

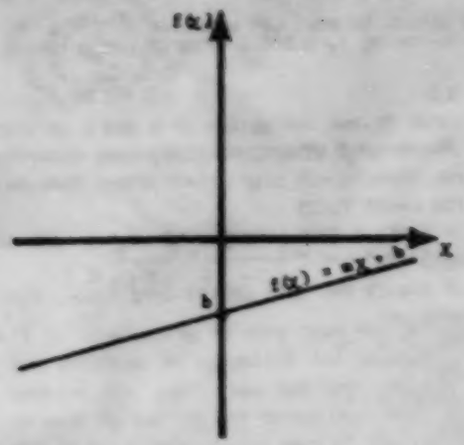
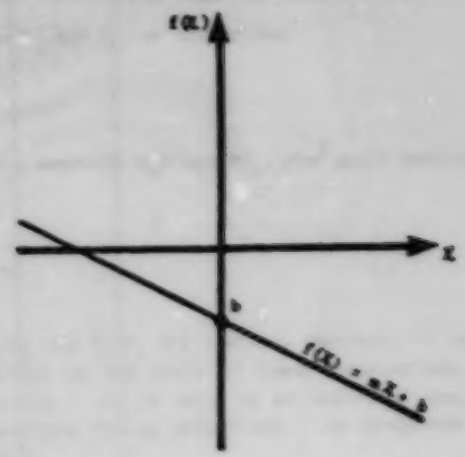
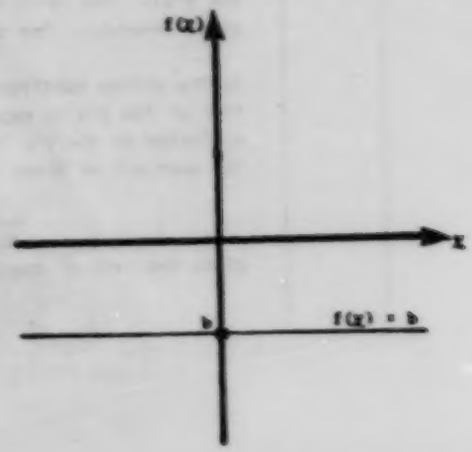
If $a > 0$, $b > 0$



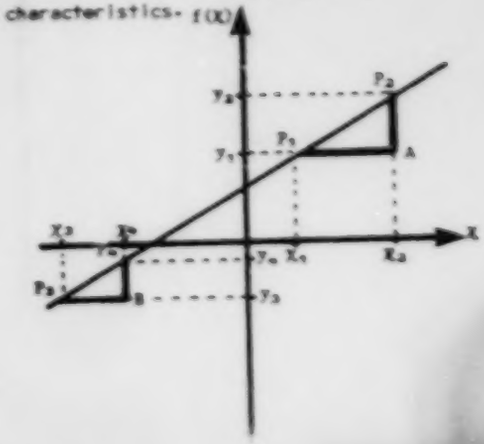
If $a < 0$, $b > 0$



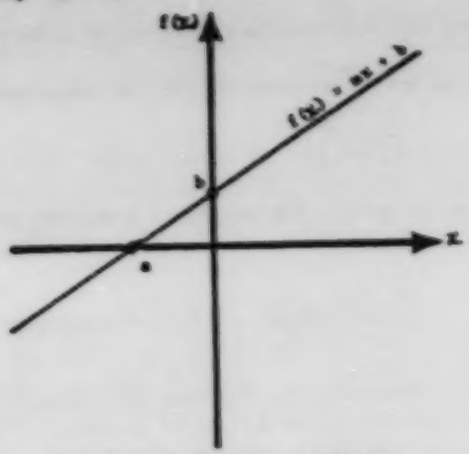
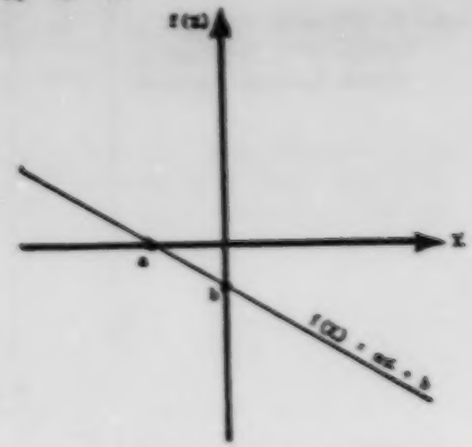
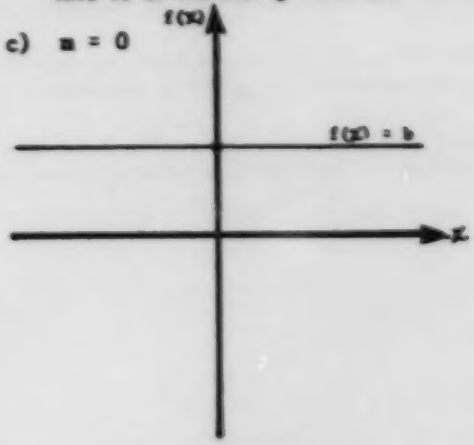
If $a = 0$, $b > 0$

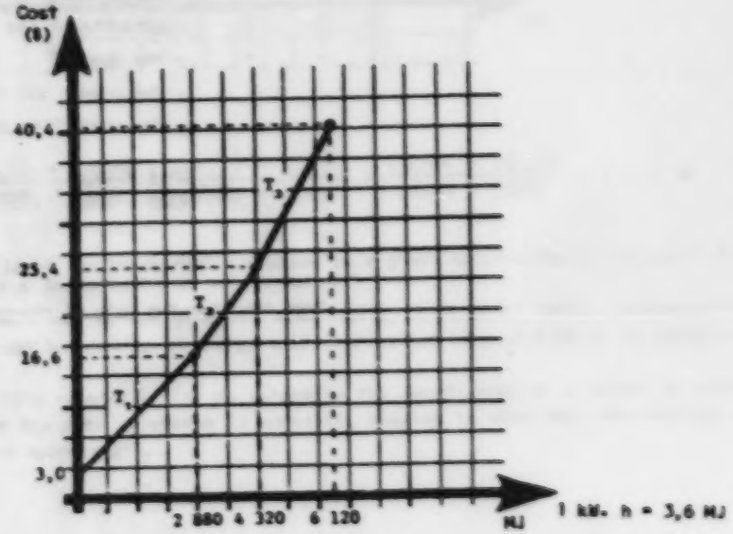
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.3 To GRAPH a constant or affine function (cont'd).	IV	If $a > 0$, $b < 0$  If $a < 0$, $b < 0$  If $a = 0$, $b < 0$ 

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.3 To GRAPH a constant or affine function (cont'd).	IV	- These last graphs should be constructed on the basis of numerical values associated with the parameters m and b. The conclusions to be reached by the pupil are identified here. - The following is a BASIC program for graphing a function. 10 CLS 20 INPUT "Enter the values of m and b in f(X)=mX+b."; M, B 30 LPRINT CHR$ (31); CHR$ (27); CHR$ (14); CHR$ (27); CHR$ (56); 40 FOR Y=25 TO -25 STEP -1 50 FOR X=-25 TO 25 60 IF Y=M*X+B THEN LPRINT CHR$ (167); ELSE GOTO 80 70 GOTO 110 80 IF X=0 OR Y=0 THEN LPRINT CHR$ (250); ELSE GOTO 100 90 GOTO 110 100 LPRINT " "; 110 NEXT X 120 LPRINT " "; 130 NEXT Y 140 LPRINT CHR$ (27); CHR$ (15); CHR$ (30); CHR$ (27); CHR$ (54)
1.3.4 To DETERMINE the rate of change of a constant or affine function.	IV	- When studying affine functions, "rate of change" and "slope" are discussed in similar terms. However, "rate of change" is a more dynamic concept. Indeed, the rate of change is determined by the quotient of two variations (variation of the x's and variation of the y's). The ratio of the variations of the x's and the y's, however, is constant in a given interval. The following comments highlight these characteristics. - In the affine function $f(x) = mx + b$, the variation of the x's is expressed by $x_2 - x_1$, and the variation of the y's is given by $y_2 - y_1$. The quotient of these two variations: $$\frac{y_2 - y_1}{x_2 - x_1}$$ gives the rate of change in the form of a ratio.

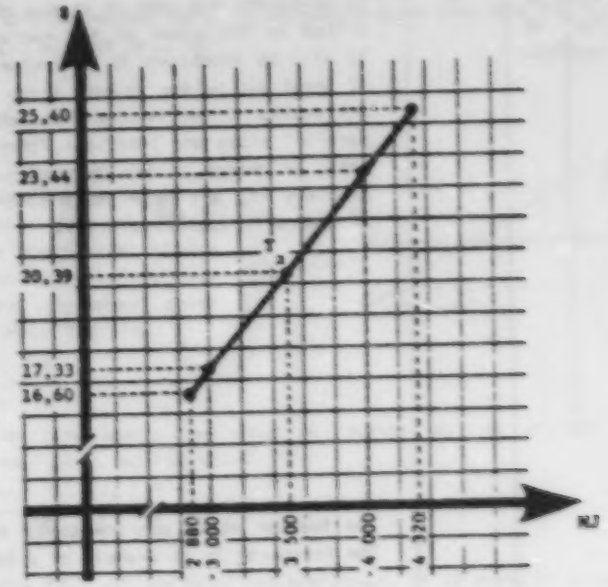


INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.4 To DETERMINE the rate of change of a constant or affine function (cont'd).	IV	- Draw the pupil's attention to the triangle P_1AP_2 formed on the graph. - If we try to evaluate the rate of change between P_3 and P_4, we can write: $\frac{Y_4 - Y_3}{X_4 - X_3}$ When we measure the segments P_1A and AP_2 and the segments P_3B and BP_4, the pupil should be made aware that: $\frac{Y_2 - Y_1}{X_2 - X_1} = \frac{Y_4 - Y_3}{X_4 - X_3} = \ldots =$ a constant $= m$. - It is interesting to note that the triangles P_1AP_2 and P_3BP_4 are similar. However, it is not possible to establish the preceding proportion on the basis of these observations, because the pupil has not yet studied similarity. It is only in an empirical way, through measuring and computing, that this proportion can be determined. The calculator can be used to evaluate each of the ratios quickly. - Help the pupil to discover that the rate of change always corresponds to the coefficient of the variable X in an affine function of the form $f(x) = mX + b$. - It is also important to have the pupil note the link between the rate of change and the direction of the line associated with the function. There are three possible situations:

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.4 To DETERMINE the rate of change of a constant or affine function (cont'd).	IV	a) $m > 0$  The rate of change is positive. This is an increasing function.
		b) $m < 0$  The rate of change is negative. This is a decreasing function.
		c) $m = 0$  The rate of change is zero. This is a constant function.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.4 To DETERMINE the rate of change of a constant or affine function (cont'd).	IV	- Point out to the pupil that a line which is parallel to the y-axis does not represent a function. Take this opportunity to discuss division by zero and the representation of a line if the denominator of its rate of change tends towards zero. - The pupil should learn to use the rate of change and the graph associated with it in order to have a better understanding of the situation which generates them. This skill is highlighted in the example of temperatures given in the note about the terminal objective. - The expression "rate of change" is used frequently in the physical sciences. It might be interesting to have the pupil deal with situations pertaining to business, the sciences, statistics, etc. A number of situations dealing with rate of change can be found in the supplementary notes. In this guide, "rate of change" will be shown on graphs as T. - The following example illustrates how electricity rates are set. 

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.4 To DETERMINE the rate of change of a constant or affine function (cont'd).	IV	To find the rate of change of each function, it is necessary to perform such computations as the following: $T_1 : \frac{16,6 - 3}{2880 - 0} = 0,004\ 722\ 22$ $T_2 : \frac{25,4 - 16,6}{4320 - 2880} = 0,006\ 111\ 11$ $T_3 : \frac{40,4 - 25,4}{6120 - 4320} = 0,008\ 333\ 33$ This also provides an opportunity to find several sequences for calculating expressions of this sort, using, for example: • parentheses, • parentheses and the command [1/x], • the memory, • the memory and the command [1/x], • the properties of fractions $\frac{25,4 - 16,6}{4320 - 2880} = \frac{25,4}{4320 - 2880} - \frac{16,6}{4320 - 2880}$ - Based on the coordinates of a number of points, it is possible to ascertain that for a given function the rate of change is constant.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.4 To DETERMINE the rate of change of a constant or affine function (cont'd).	IV	

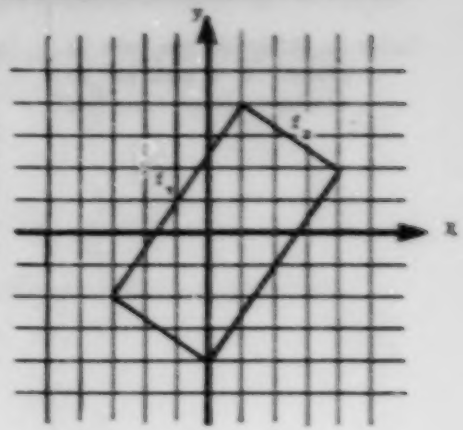
$$\frac{17,33 - 16,60}{3000 - 2880} = \frac{20,39 - 17,33}{3500 - 3000} = \frac{23,44 - 20,39}{4000 - 3500} = \ldots = m$$

The same experiment can be carried out based on a graph constructed by the pupil himself.

Hydro-Québec will not be changing its electricity rates until 1987. Consequently, the kilowatt-hour can be used as an energy unit if required in a problem or an example.

It is the pupil's responsibility to determine the coordinates of a number of points and to verify that the rate of change is constant, keeping in mind that the readings of the coordinates are approximate.

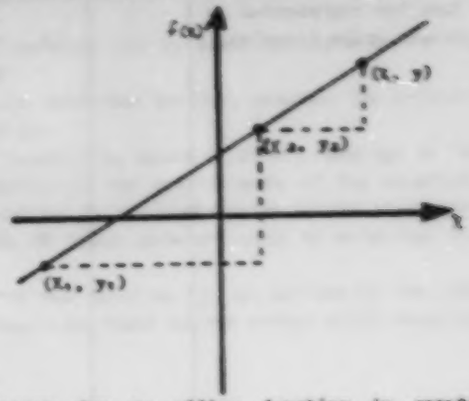
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.4 To DETERMINE the rate of change of a constant or affine function (cont'd).	IV	To calculate the rate of change of each function described by the sides of the rectangle in this graph, it will be necessary to determine the coordinates of at least two points on each side. The teacher should ask pupils to verify that the rates of change of two parallel sides are identical, that the product of the rates of change of two perpendicular sides is approximately -1, and that the product of the four rates of change is approximately 1.



- The teacher might ask the pupil to write a program for evaluating this rate of change. The following is an example of a BASIC program written for this particular purpose:

```
10 CLS
20 INPUT "Enter the values of m and b associated with the function f(X)=mX+b"; M, B
30 C=-10
40 D=10
50 E=-10*M+B
60 F=10*M+B
70 V=(E-F)/(C-D)
80 PRINT "The rate of change of f(X)="; M; "X+"; B; "is"; V
90 END
```

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.5 To WRITE the equation of a function represented by a linear graph.	IV	- Using the graph reproduced here, it is possible to evaluate the rate of change based on the points (x_1, y_1) and (x_2, y_2)



$$\frac{y_2 - y_1}{x_2 - x_1}$$

It is also possible to determine this particular rate of change based on the coordinates of one of the two points, in this case (x_1, y_1) and any other point (x, y) located on the line

$$\frac{y_1 - y}{x_1 - x}$$

Since, as we have seen, the rate of change for an affine function is constant (cf. Objective 1.3.4), we can write:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y}{x_1 - x} \qquad (1)$$

Since the pairs (x_1, y_1) and (x_2, y_2) are known, it is easy to calculate the slope of the line that crosses these points:

$$\frac{y_2 - y_1}{x_2 - x_1} = m.$$

By substituting in (1) we can write:

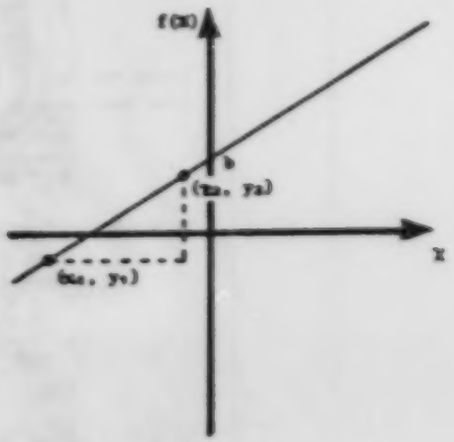
$$\frac{y_1 - y}{x_1 - x} = m. \qquad (2)$$

If the y-intercept b is known, it is possible to write:
$(x_1, y_1) = (0, b)$.

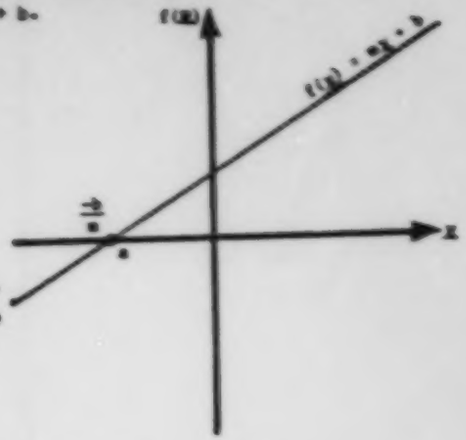
By substituting in (2), we have:

$$\frac{b - y}{-x} = m.$$

By means of simple algebraic transformations, we discover the equation $y = mx + b$. (3)

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.5 To WRITE the equation of a function represented by a linear graph (cont'd).	IV	- This demonstration can be applied in three different situations: 1. Finding the equation associated with a given function if the coordinates of two points are known. - In this case, we use equation (1), substituting the coordinates of the pairs (x_1, y_1) and (x_2, y_2) obtained by direct reading from the appropriately selected points P_1 and P_2. 2. Finding the equation associated with a function if the slope and one point are known. - We use equation (2), substituting the slope and the coordinates of the pair (x_1, y_1) obtained by direct reading from the appropriately selected point P_1. - The slope is evaluated by causing a variation of a given value such as the abscisse of a point and by evaluating the variation brought about at the ordinate. The quotient of these variations $\frac{y_2 - y_1}{x_2 - x_1}$ gives the slope m. 

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.5 To WRITE the equation of a function represented by a linear graph (cont'd).	IV	3. Finding the equation associated with a function if the slope and the y-intercept are known. - Since the y-intercept can be expressed by means of a pair of the form (0, b), it is possible to use equation (2). - It is more convenient, however, to use equation (3) by substituting directly the slope m and the y-intercept b. - The slope is found by using the technique described earlier, whereas the y-intercept is found by direct reading on the graph. - Finding the equation associated with a function by means of direct readings on the graph can result in an imprecise evaluation of the coefficients of the equation. When certain points on the line are identified by the coordinates of the associated pairs, the results will be more precise if these data are used to establish the equation. - For example, the equation associated with the function T_2, as defined by the last graph in the note on Objective 1.3.4, could be found by the method which requires the use of equation (1). Let $(x_1, y_1) = (2880, 16,60)$ $(x_2, y_2) = (4320, 25,40)$ Then (1) gives: $$\frac{25,40 - 16,60}{4320 - 2880} = \frac{16,60 - y}{2880 - x}$$ $$0,006\ 111\ 11 = \frac{16,60 - y}{2880 - x}$$ $$17,60 - 0,006\ 111\ 11\,x = 16,60 - y$$ $$y = 0,006\ 111\ 11\,x - 1$$ Therefore $T_2(x) = 0,006\ 111\ 11\,x - 1$ - In this example the slope is almost zero (m = 0,006 111 11), and yet the line representing T_2 does not have the gentle slope we might expect. This situation is due to the change in scale which affects the X- and y-axes in the graph associated with this function. The notes accompanying Intermediate Objective 3.2.3 show the effects of a change in scale on a given graph.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.6 To CALCULATE the zero(s) of a constant or affine function.	IV	- To find the zero(s) of a function is to find the value of the variable "x" when f(x) is equal to zero. Consider the affine function $f(x) = ax + b$. Then if: $f(x) = 0$ we have: $ax + b = 0$, $ax = -b$, $x = \frac{-b}{a}$, given that $a \neq 0$.  - The zero(s) of an affine function, therefore, is (are) given by $\frac{-b}{a}$, and is (are) found at the intersection of the x-axis and the line associated with the function. - This value (the zero) is also called "the x-intercept" or the solution of the equation $ax + b = 0$ associated with the function. The x-intercept is identified by "a". - Here again the calculator can be used. Consider the function $f(x) = 0,38x - 13$. We want to find out for what numerical value the procedure [×] 0,38 [=] [−] 13 [=] will display 0. By finding the inverse procedure [+] 13 [=] [÷] 0,38 [=] and starting from 0, we find 34,21 to the nearest hundredth. Using a table we arrive at the same answer by the framing method:

x	0,38x - 13
10	-9,2
20	-5,4
30	-1,6
40	2,2

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.3.6 To CALCULATE the zero(s) of a constant or affine function (cont'd).	IV	We start over, using values between 30 and 40; then we enter 34 and 35, and so on. - In the case of T_2 (final notes on Objective 1.3.5), if $T_2 = 0$, we have $0{,}006\ 111\ 11X - 1 = 0.$ With the aid of a calculator and using the equation $X = \frac{-b}{a}$ we find the zero of T_2: $X = \frac{1}{0{,}006\ 111\ 11}$ $X = 163,\ 636\ 363.$ - In this case, the interpretation "an energy consumption of 163,636 363 MJ would cost $0,00" is meaningless since a different rate, which is defined by the function T_1, is applied to an energy consumption of less than 2880 MJ. This is another situation in which the mathematical model used for defining T_2 does not suit reality. This model is: $T_2(X) = 0{,}006\ 111\ 11X - 1$. To make it fit reality, it is necessary to reduce the domain of T_2 to the interval [2880, 43201. - The pupil could write a program similar to the one presented in Objective 1.2.3. Here is an example of such a program written in BASIC: 10 CLS 20 INPUT "Enter the values associated with a and b in f(X)=aX+b."; M, B 30 IF B=0 THEN IF M=0 THEN PRINT "There are countless zeros." ELSE PRINT "There is no zero." 40 IF M=0 THEN STOP ELSE X=B/M 50 PRINT X; "is the zero of this function." 60 END

TERMINAL OBJECTIVE 1.4 TO SOLVE problems arising from situations involving a system of equations of the first degree in two variables.

The teaching of equation systems shows that it is relatively easy to mathematize many situations by means of propositional forms with two or more variables. However, selecting relevant and realistic situations is often a difficult task. If the pupil is to recognize the usefulness of this subject, an effort should be made to select situations that pertain to business, consumption, recreational activities, and other everyday life concerns.

A system of equations necessarily involves the intersection of the solution sets associated with these equations. There is a variety of algebraic methods that can be used to find the intersection of the solution sets associated with each propositional form. While the guide does not emphasize any one particular method, it suggests a number of them which the teacher is encouraged to present to the pupil and from which the latter will select. What is important is that the pupil should know how to find the solution set of the system of equations; the method he uses for that purpose is irrelevant.

It should be pointed out that the mathematical knowledge and skills used here for solving problems translated into a system of equations can be used for solving systems of equations per se.

The terminal objective specifies the number of real variables of which systems of equations are made. However, progressing from solving equations of the first degree in two real variables to studying systems with more than two real variables is a natural transition. The transformation rules remain valid; only the amount of time it takes to apply them changes. The pupil's exposure to equations with more than two variables should be to his benefit. He is given the opportunity to develop his ability to mathematize, to

explore new avenues, to seek quick and effective methods. He is encouraged to use matrices which, like detached coefficients or Cramer's rule, make possible the mathematical solution of many situations.

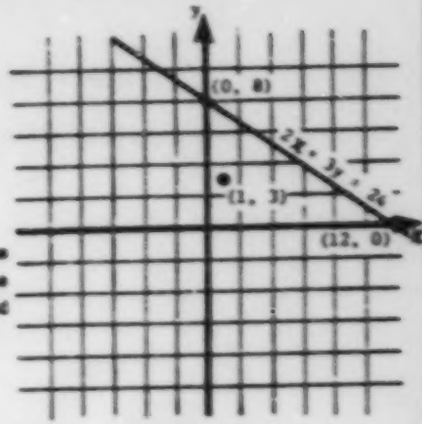
The use of matrices for solving systems of equations should not seem so strange. In fact, in the new technologies the use of matrices is quite common, and the detached-coefficient method, as well as Cramer's rule, are well-known variants of matrical computation. Matrices are very effective tools for mathematizing real situations. The notes pertaining to a number of intermediate objectives and the supplementary teaching notes provide interesting information on this subject.

The calculator also is a very effective tool for solving systems of equations by the repetitive method. The use of this method provides an opportunity for the teacher to introduce the pupil to the concept of framing (by default or by excess) and, in a very intuitive manner, to the concept of limit. Graphic representations of the different phenomena studied should prove very helpful to the pupil in his exploration of these concepts.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.1 To REPRESENT a given situation by a system of equations or inequations of the first degree in two variables.	IV	- Constructing a system is relatively simple. It is only necessary to formulate a number of hypotheses and to devise propositional forms based on their wording. Example: One Saturday at the movies Paul counts 312 people in the theater, and Pauline discovers that there are twice as many children as adults. Solution: Let x be the number of children and y the number of adults. $x + y = 312$: "Paul counts 312 people;" $x = 2y$: "Pauline discovers that there are twice as many children as adults." - It may, however, be difficult and even impossible to translate a given situation into a system of propositional forms. Example: One Saturday at the movies Paul notices that the children and adults present forms a group of 312 people. On Sunday Pauline counts twice as many children as adults in the theater. For x children and y adults, the pupil is tempted to write: $x + y = 312$ (1) $x = 2y$ (2) Equation (1) describes the situation as it was on Saturday and Equation (2) as it was on Sunday. However, these two equations cannot make up a system because the data do not bear on the same group of people. (This is not specified in the statement of the problem). - Examples of this sort should encourage pupils to investigate and discuss problems in greater detail, which should contribute to the development of their critical abilities. - Equations that make up a system must describe different aspects of the same situation. - ENRICHMENT: Use situations that require more than two variables. Point out to the pupil that a system of equations has a finite number of solutions only if it is composed of as many separate non-equivalent equations as there are variables.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.2 To VERIFY if an ordered pair is the solution set of a given equation.	IV	- Consider the expression: $2x + 3y = 24$. If a pair is given, all that has to be done is to make a substitution in order to establish the truth set (value) of the proposition. If we have the ordered pair (1, 3), $2x + 3y = 24$ becomes $2(1) + 3(3) \neq 24$, which means that the pair does not meet the condition. If the purpose is to determine ordered pairs that are appropriate for the expression, it is possible to construct a grid and to complete it so as to obtain pairs that are solutions of the given propositional form.

x	y	(x, y)
0	8	(0, 8)
12	0	(12, 0)



- A graph can also be used to verify if an ordered pair is the solution set of a given equation. In the above case it can be observed that the point (1, 3) is not on the line associated with the equation $2x + 3y = 24$.

- It is possible to establish a procedure that can be used to verify if an ordered pair is the solution set of a given equation.
For example, the procedure

x [x] 2 [=] [STO]

y [x] 3 [=] [+] [RCL] [=] [+] 24 [+/-] [=]

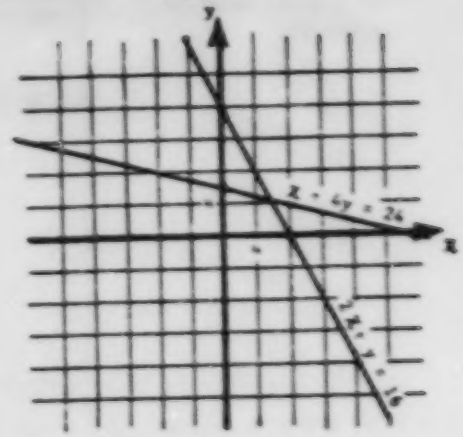
is associated with the equation
$2x + 3y = 24$.

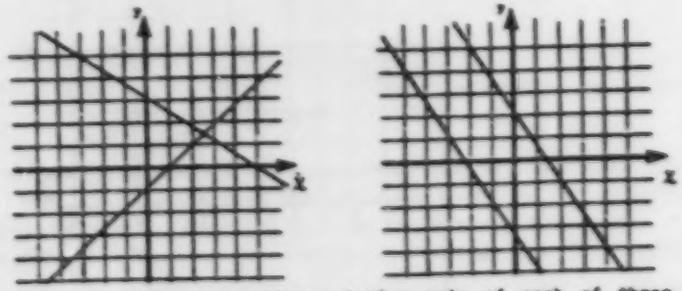
When we replace x and y by the coordinates of an ordered pair, the result will be 0 or approximately 0 if the pair is the solution set of the equation. It should be pointed out that it is necessary to store $2x$ in the calculator's memory if the machine does not perform operations in the order in which they are entered.

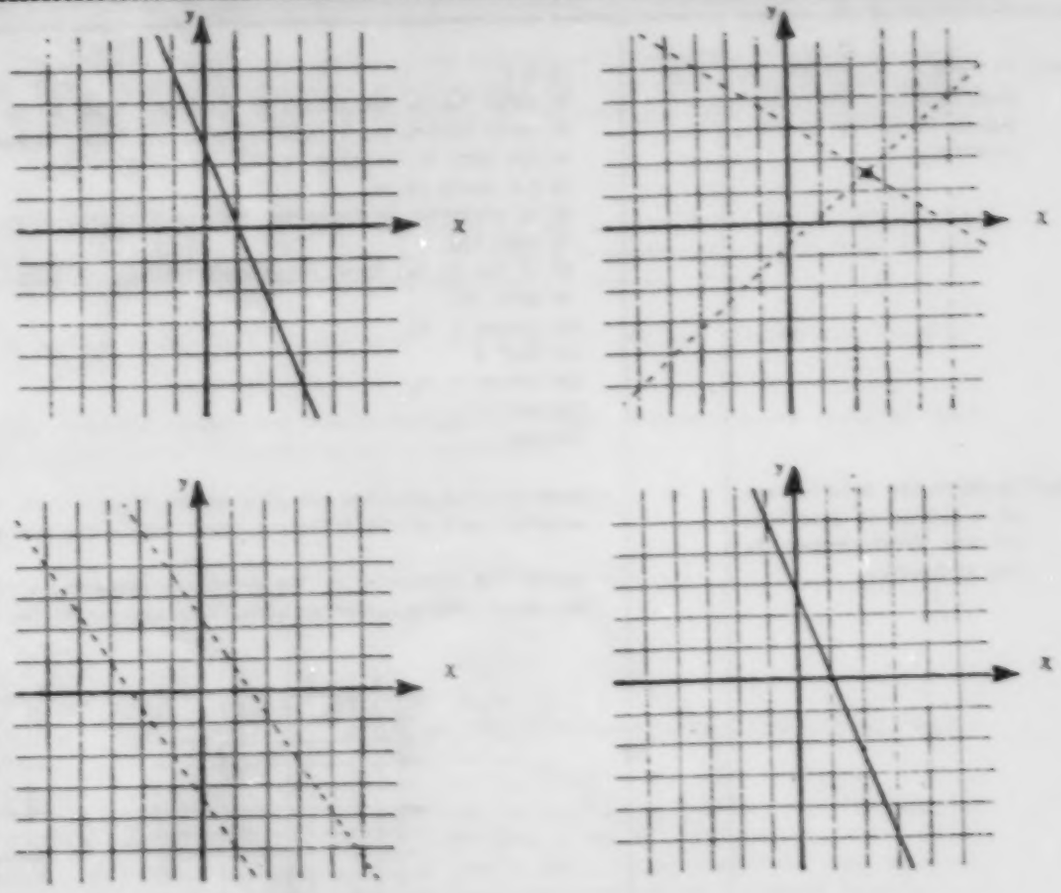
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.2 To VERIFY if an ordered pair is the solution set of a given equation (cont'd).	IV	On a calculator that performs operations in the order in which they are entered, the procedure would be: 2 [×] x [+] 3 [×] y [−] 24 [=] - A short computer program can also be used to perform the required verification. Such a program could be written in BASIC as follows: 10 CLS 20 INPUT "Enter the values associated with the coefficients of Ax+By=C"; A, B, C 30 INPUT "Enter the elements of the ordered pair (x, y)."; X, Y 40 IF A*X+B*Y=C THEN PRINT "This ordered pair is the solution set of the equation." ELSE PRINT "This ordered pair is not the solution of the equation." 50 END
1.4.3 To GRAPH a system of equations of the first degree in two variables.	IV	- The pupil has to construct a line corresponding to each equation. Example: Consider the system $x + 4y = 24$ $2x + y = 16$ The pupil should construct a grid for each equation (a) $x + 4y = 24$ (b) $2x + y = 16$

x	y
0	6
24	0
8	4
12	3

x	y
0	16
4	8
8	0
5	6

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.3 To GRAPH a system of equations of the first degree in two variables (cont'd).	IV	- It is advisable to point out the alignment of the points and the generalization that can be drawn by examining the pattern that emerges from this graphing process. - The graph makes it possible to visualize either overlapping or distinct parallel lines or intersecting lines. - This method, however, often gives only approximative results. The pupil should be made aware of this fact. - It is possible to establish procedures by means of which the graph associated with the system can be constructed. The procedures associated with the above system would be:  [+/-] [+] 24 [=] [÷] 4 [=] for $y = \frac{-x + 24}{4}$ [x] 2 [+/-] [=] [+] 16 [=] for $y = -2x + 16$ These procedures generate ordered pairs. By introducing a certain value for x at the beginning, each procedure evaluates the corresponding value for y. All that remains to be done is to transfer the ordered pairs produced by each procedure to the Cartesian plane and to draw the lines associated with each equation. - It should be noted that the set of lines drawn in the same plane represents the system of equations. In general, a system of equations is defined as a finite set of equations. - The following program in BASIC makes it possible to represent a system of equations in the Cartesian plane.

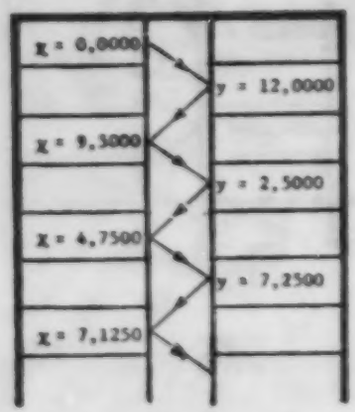
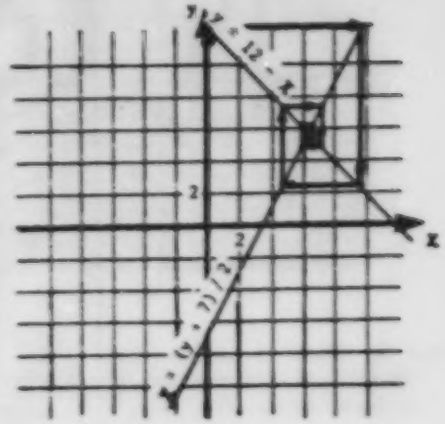
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.3 To GRAPH a system of equations of the first degree in two variables (cont'd).	IV	10 CLS 20 INPUT "Write the values of A, B, and C in the equation Ax+By=C"; A, B, C 30 INPUT "Write the values of D, E, and F in the equation Dx+Ey=F"; D, E, F 40 FOR Y=25 TO -25 STEP -1 50 FOR X=-25 TO 25 60 IF A*X+B*Y=C OR D*X+E*Y=F THEN LPRINT CHR$ (167); ELSE GOTO 80 70 GOTO 110 80 IF X=0 OR Y=0 THEN LPRINT CHR$ (250); ELSE GOTO 100 90 GOTO 110 100 LPRINT " "; 110 NEXT X 120 LPRINT " "; 130 NEXT Y 140 END
1.4.4 To GRAPH the solution set of a system of equations of the first degree in two variables.	IV	- Generally the solution set of a system of equations is defined as the intersection of the solution sets of the different equations in the system. - Within the framework of the exercises connected with the preceding objective, the pupil has dealt with a number of situations similar to the following:  The respective solution sets of each of these situations are shown on the following page. (The dotted lines mean that the elements of these lines do not belong to the solution set, except for the point of intersection in the first example).

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.4 To GRAPH the solution set of a system of equations of the first degree in two variables (cont'd).	IV	 - As the solution set of a system of equations is either a point, an empty set, or the set of ordered pairs associated with the overlapping lines, it cannot be adequately represented by the first series of graphs.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.4 To GRAPH the solution set of a system of equations of the first degree in two variables (cont'd).	IV	- It is possible, however, to represent the solution set of a system of equations by means of the first series of graphs by adding colors. But it is still preferable to represent the solution set of a system of equations by the technique used in the second set of graphs. - Later, in Objective 1.4.6, it will be explained how a graph can be used to gain a better understanding of the repetitive method for finding a solution set. Additional information on this method is provided in the supplementary teaching notes.
1.4.5 Given a system of equations of the first degree in two variables, TO IDENTIFY an equivalent system.	IV	- Two systems of equations are equivalent if and only if they have the same solution set. - These are the elementary transformations or linear combinations by means of which equivalent systems of equations can be constructed: . Multiplying or dividing an equation by a scalar: $x + 4y = 24$ (1) $2x + y = 16$ (2) is equivalent to $2x + 8y = 48$ (3)' $2x + y = 16$ (4) . Adding or subtracting two equations: By subtracting (4) from (3) we obtain a new equation equivalent to the first: $7y = 32$ (5)' $2x + y = 16$ (6) - A simple look at the elementary transformations that are used should make it clear that it is not necessary to write down the variables x and y. It is better, therefore, to retain only the coefficients and to use a matrical representation (the detached-coefficient method). These are the matrices associated with the preceding three systems: $\begin{bmatrix} 1 & 4 & 24 \\ 2 & 1 & 16 \end{bmatrix} \quad \begin{bmatrix} 2 & 8 & 48 \\ 2 & 1 & 16 \end{bmatrix} \quad \begin{bmatrix} 0 & 7 & 32 \\ 2 & 1 & 16 \end{bmatrix}$

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.5 Given a system of equations of the first degree in two variables, TO IDENTIFY an equivalent system (cont'd).	IV	- By using elementary transformations for finding the simplest equivalent system of equations, it is possible to obtain the following intermediate systems of equations: • By multiplying (6) by 7: $7y = 32$ (7) $14x + 7y = 112$ (8)* $\begin{bmatrix} 0 & 7 & 32 \\ 14 & 7 & 112 \end{bmatrix}$ • By subtracting (7) from (8): $7y = 32$ (9) $14x = 80$ (10)* $\begin{bmatrix} 0 & 7 & 32 \\ 14 & 0 & 80 \end{bmatrix}$ • By dividing (9) by 7 and (10) by 14: $y = \frac{32}{7}$ (11)* $x = \frac{40}{7}$ (12)* $\begin{bmatrix} 0 & 1 & \frac{32}{7} \\ 1 & 0 & \frac{40}{7} \end{bmatrix}$ - The asterisk indicates that the equation has been modified in relation to the preceding system. - To solve a system of equations, we usually look for equivalent systems. For example, the system $2x - y = 2$ $4x - 5y = 5$ is equivalent to $2x - y = 2$ $2x - 4y = 3$

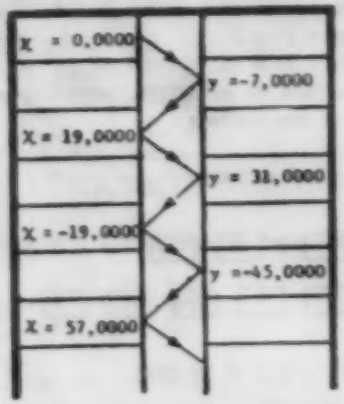
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.5 Given a system of equations of the first degree in two variables, TO IDENTIFY an equivalent system (cont'd).	IV	- When the pupil is asked to construct the Cartesian graph of these two systems, he should be made aware that in the respective graph associated with each equivalent system the lines have the same intersecting point. - It can be noted in the first example that the simplest equivalent system $y = \frac{32}{7}$ $x = \frac{40}{7}$ makes it possible to arrive at the solution of the system: $(x, y) = (\frac{40}{7}, \frac{32}{7})$.
1.4.6 TO SOLVE a system of equations of the first degree in two variables.	IV	- To solve a system of equations, it is sufficient to search for equivalent systems until the simplest equivalent system is found. The detached-coefficient method is very effective for this purpose. (Cf. Objective 1.4.5). - More traditional methods, such as reduction, comparison, substitution, and Cramer's rule, can also be used for this purpose. The supplementary teaching notes present an example of each of these methods for solving systems of equations. - It should be interesting to solve a system with the aid of a calculator, using the repetitive method. Such a method should make it possible to "frame" the solution of the system by providing a satisfactory approximation. Consider, for example, the system $x + y = 12$ $2x - y = 7$ which is equivalent to $y = -x + 12$ (1) $x = \frac{y + 7}{2}$ (2)

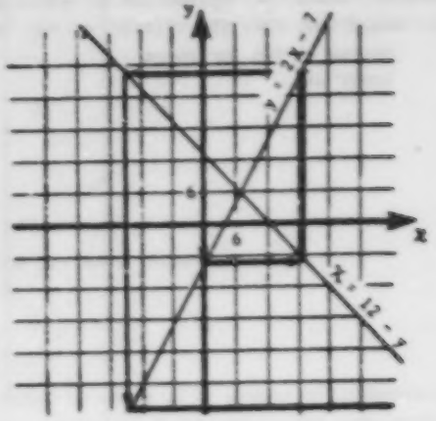
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.6 TO SOLVE a system of equations of the first degree in two variables (cont'd).	IV	or to $x = -y + 12$ (3) $y = 2x - 7$ (4) Taking 0 as the initial value of x ($x = 0$), we alternately evaluate (1) and (2) a number of times. Considering only the first four decimals, we obtain the following results:   On the table we note that x moves toward $6,\overline{3}$ y moves toward $5,\overline{6}$ At the tenth step, x = 6,3457 and y = 5,6543, if we accept an answer to the nearest tenth (0, 1). It should be noted that if we posit x = 0 in (4) and alternately evaluate (4) and (3) a number of times, we obtain the following table:

INTERMEDIATE OBJECTIVES	LEVELS	NOTES

1.4.6 TO SOLVE a system of equations of the first degree in two variables (cont'd).

IV





We move away from the point of intersection. In fact, at the tenth step, X = 3249 and y = 6491. Such a situation leads nowhere and the pupil has to learn to recognize it so as to be able to reject it.

- The procedures that are used here are as follows:

+/- + 12 STO	for $y = -X + 12$	(1)
(RCL + 7) ÷ 2 STO	for $y = \frac{y + 7}{2}$	(2)
+/- + 12 STO	for $X = -y + 12$	(3)
2 x RCL - 7 STO	for $y = 2X - 7$	(4)

- The teacher should explore with the pupils those cases where the lines are parallel or perpendicular to each other, as well as the situations in which one or both lines of the system are parallel to the axes. This subject is covered briefly in the supplementary teaching notes.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.6 TO SOLVE a system of equations of the first degree in two variables (cont'd).	IV	- It is also possible to find the solution of the system by substituting values with the help of the calculator. For example, for the system $2x - y = 2$ $4x - 5y = -5$ it is possible to establish certain procedures for verifying if the ordered pair is the solution of the system. For $2x - y = 2$ we have x [x] 2 [=] [STO] y [+/-] [+] [RCL] [=] [-] 2 [=] and for $4x - 5y = -5$ we have x [x] 4 [=] [STO] y [x] 5 [+/-] [=] [+] [RCL] [=] [+] 5 [=] If we replace x and y by the coordinates of an ordered pair, it is possible to determine whether the pair is the solution of the system when the result of the two procedures is 0 or approximately 0. It is necessary to store $2x$ and $4x$ in the calculator's memory if the machine that is used does not perform operations in the order in which they are entered. With a calculator that performs operations in the order in which they are entered, the following procedures might be used: 2 [x] x [-] y [-] 2 [=] 4 [x] x [-] 5 [x] y [+] 5 [=]

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.6 TO SOLVE a system of equations of the first degree in two variables (cont'd).	IV	- There is another very effective way of solving a system of equations of the first degree in two variables. The calculator or the computer can be used to evaluate the following formulas: $x = \frac{C_1B_2 - B_1C_2}{A_1B_2 - B_1A_2}$ $y = \frac{A_1C_2 - C_1A_2}{A_1B_2 - B_1A_2}$ with $A_1B_2 \neq B_1A_2$. These two formulas come from the system of equations $A_1x + B_1y = C_1$ $A_2x + B_2y = C_2$ from which the variables x and y have been isolated by means of a number of elementary algebraic transformations. Cramer's rule (method), which is discussed in the supplementary teaching notes, derives from these formulas. - The following two sequences make it possible to find the respective value of x and y with the aid of a calculator that performs operations in the order in which they are entered: [(] C_1 [×] B_2 [−] B_1 [×] C_2 [)] [÷] [(] A_1 [×] B_2 [−] B_1 [×] A_2 [)] [=] [(] A_1 [×] C_2 [−] C_1 [×] A_2 [)] [÷] [(] A_1 [×] B_2 [−] B_1 [×] A_2 [)] [=] - The following BASIC program can be used to help find the solution of a system of equations by discussing the different aspects of that solution.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.4.5 TO SOLVE a system of equations of the first degree in two variables (cont'd).	IV	10 CLS 20 INPUT "Write the values of A, B, and C in the equation AX+By=C"; A, B, C 30 INPUT "Write the values of D, E, and F in the equation DX+Ey=F"; D, E, F 40 IF A*E=B*D AND B*F=C*E THEN PRINT "The lines are overlapping parallels and the system has a countless number of solutions." 50 IF A*E=B*D AND B*F< >C*3 THEN PRINT "The lines are separate parallels and the system has no solution." ELSE GOTO 70 60 STOP 70 X=(C*E-B*F)/(A*E-B*D) 80 Y=(A*F-C*D)/(A*E-B*D) 90 PRINT "The lines intersect each other and the solution of the system is ("; X; "; "; Y; ")." 100 END

TERMINAL OBJECTIVE 1.5 TO SOLVE problems arising from situations involving a system of inequations of the first degree in two real variables.

Up to this point in secondary school, inequations of the first degree in one or two variables as well as their representation in the Cartesian plane have been studied without any concern for their concrete applications. The only justification for treating this subject in such a theoretical way at the secondary school level seems to be the importance of having a sound knowledge of the axioms of order and of being able to apply them correctly. Nevertheless, the graphic representation of a system of inequations can have very practical applications and the creation of a system of inequations can make it possible to mathematize certain situations very easily. A very large number of situations can be mathematized by means of inequations and, in today's technological society, inequations have as many applications as do equations.

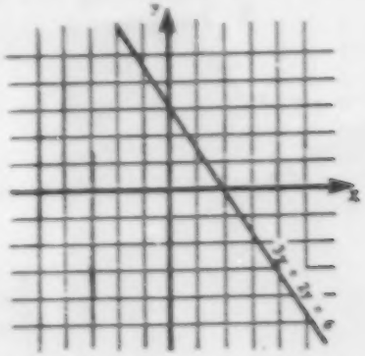
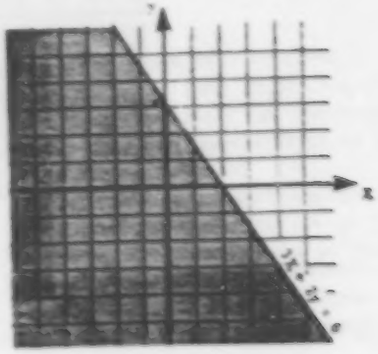
In situations which can be mathematized by means of a system of inequations, the goal is very often to optimize the solutions that are found. In this regard, this objective is intended as a brief but practical introduction to linear programming. What is involved here is indeed programming, because the sequence of operations that makes it possible to find the optimal solution unfolds in a specific order and constitutes a program. The type of programming involved is called linear because the phenomena that are mathematized are represented by means of equations and inequations of the first degree.

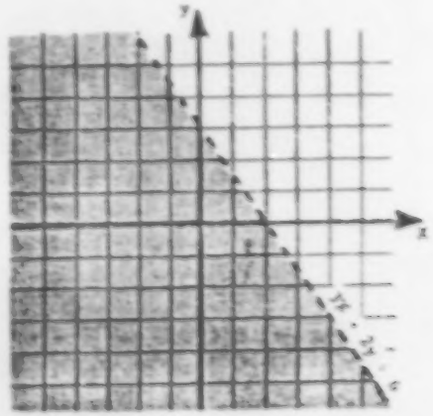
In the second cycle mathematics program for secondary school there is no question of using very complex problem-solving methods and of studying the simplex method. The method that is being suggested can be described as follows:

1. Mathematization of the situation by means of a system of inequations (construction of a model);
2. Representation of the system of inequations in the Cartesian plane (construction of the constraining polygon);
3. Identification of the function to be optimized (sliding line);
4. Search for the optimal solution by substituting values in the function to be optimized or by translating the sliding line on the constraining polygon;
5. Interpretation of the result(s).

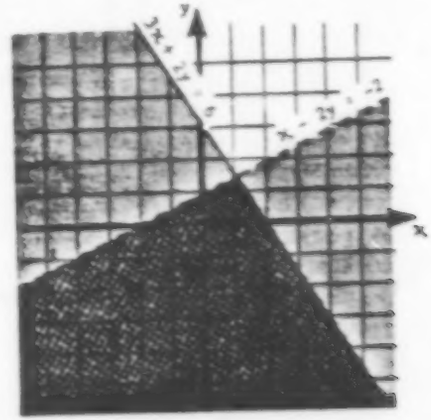
The study of equations and inequations of the first degree and of their graphic representation should ultimately enable the pupil to solve optimization problems. In this context, linear programming is a subject with a broad variety of applications. It is advisable, however, to select situations that can illustrate for the pupil, based on reduced models if necessary, the usefulness and effectiveness of mathematics as a decision-making tool. Obviously, while the situations selected have to resemble real situations in business and industry, they should be simple and interesting for the pupil.

Thanks to the careful selection of situations (agriculture, inventory check, production, food production, stocking, personnel assignment, transportation, etc.), the use of heuristics, and the systematic use of the calculator or the computer, the pupil should be able to complete the study of the subject successfully.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.1 TO VERIFY if an ordered pair is the solution set of a given system of inequations of the first degree in two variables.	IV	- Before completing this objective the pupil should first verify if an ordered pair is the solution of an inequation. The results of this exercise will tell him from what half-plane he should take the appropriate values. Here is an example that illustrates this process: Given the inequation $3x + 2y \leq 6$. The equation line $3X + 2y = 6$ is represented thus:  - The inequation $3x + 2y \leq 6$ is represented by this graph: 

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.1 TO VERIFY If an ordered pair is the solution set of a given system of inequations of the first degree in two variables (cont'd).	IV	The solution is (0, 0) since by substituting its elements in the inequation we obtain a true proposition: $3(0) + 2(0) \leq 6$ $0 \leq 6$ It is possible, therefore, to establish the following rule: If the elements of an ordered pair which are substituted in an inequation transform that inequation into a true proposition, this ordered pair is the solution of the inequation. - The set of points located on the equation line associated with the inequation constitutes the boundary of the half-plane. A straight line indicates that the boundary belongs to the relation, whereas a broken line indicates that the boundary does not belong to the relation. This is the graph of the inequation $3x + 2y < 6$:  - Finding the solution set of a system of inequations by an algebraic method is a long and complex process. At this point it is sufficient to construct a graph of the system of inequations in order to visualize the solution set.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.1 TO VERIFY if an ordered pair is the solution set of a given system of inequations of the first degree in two variables (cont'd).	IV	- On a given graph, an ordered pair is the solution of a system of inequations if that pair is located at the intersection of the half-planes associated with the inequations of the system. Consider the system $3x + 2y \leq 6$ $x - 2y > -2$



It is obvious on the graph that (-1, -4) is the solution of the system. However, in some cases, with the ordered pair (2,27, -0,39) for example, it is difficult to be sure, because the graph is normally an approximative representation of a given mathematical reality. In such cases the substitution method is an indispensable tool.

> An ordered pair verifies a system of inequations if and only if the substitution of its elements transforms all the propositional forms of the system into true propositions.

The ordered pair (-1, -4) is the solution of the system, whereas (2,27, -0,39) is not.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES

1.5.1 TO VERIFY if an ordered pair is the solution set of a given system of inequations of the first degree in two variables (cont'd).

IV

Ordered Pairs	Substitutions	Truth Values	
(-1, -4)	$3(-1) + 2(-4) \leq 6$ $-11 \leq 6$	True	True
	$(-1) - 2(-4) > -2$ $7 > -2$	True	
(2,27, -0,39)	$3(2,27) + 2(-0,39) \leq 6$ $6,03 \leq 6$	False	False
	$(2,27) - 2(-0,39) > -2$ $3,05 > -2$	True	

1.5.2 TO GRAPH a system of inequations of the first degree in two variables.

IV

- To construct the Cartesian graph of the system of inequations

$$x - 2y \geq -5$$
$$7x - 2y \leq 13$$
$$x - 7 \geq -2$$

we perform the following tasks for each inequation in the system:

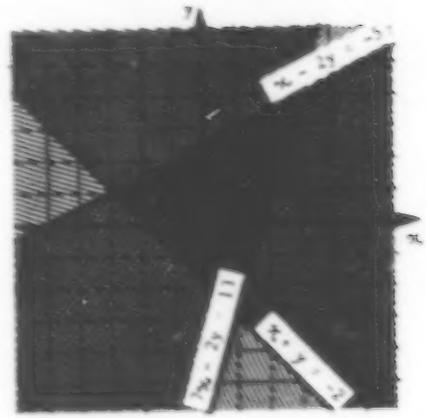
. we draw a straight line or a broken line which corresponds to the equation under construction.

. we identify the half-plane associated with that particular inequation, applying the substitution rule stated in the notes on Objective 1.5.1. It should be remembered that this rule is valid only for the points that do not belong to the boundary of the half-plane, that is, to the line. Moreover, when the line does not go through the origin it is advisable to use the ordered pair (0, 0) to carry out the substitution.

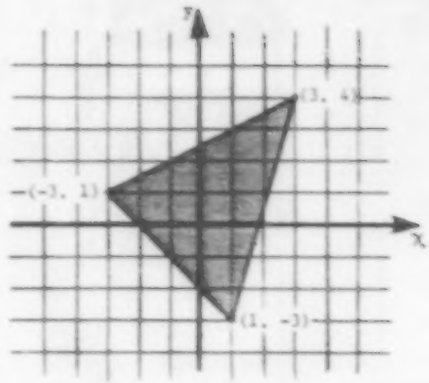
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.2 TO GRAPH a system of inequations of the first degree in two variables.	IV	

Ordered Pairs	Substitutions	Truth Values	
(0, 0)	$(0) - 2(0) \geq -5$ $0 \geq -5$	True	True
	$7(0) - 2(0) \leq 13$ $0 \leq 13$	True	
	$(0) + (0) \geq -2$ $0 \geq -2$	True	

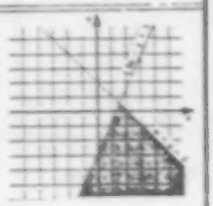
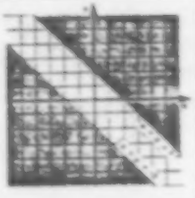
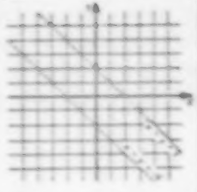
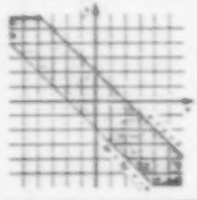
Here is the graphic representation of this particular situation:



1.5.3 TO GRAPH the solution set of a system of inequations of the first degree in two variables.	IV	- The solution set of a system of inequations always corresponds to the intersection of the solution sets defined by each element of the system. In other words, the solution set of a system of inequations corresponds to the set of ordered pairs which transforms simultaneously all the inequations of the system into true propositions.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.3 TO GRAPH the solution set of a system of inequations of the first degree in two variables (cont'd).	IV	- This is the graphic representation of the solution set associated with the system $x - 2y \geq -5$ $3x - 2y \leq 13$ $x + y \geq -2$ 

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.3 TO GRAPH the solution set of a system of inequations of the first degree in two variables (cont'd).	IV	

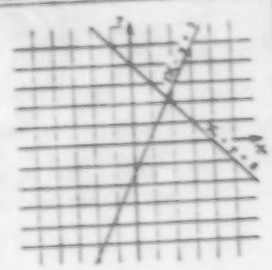
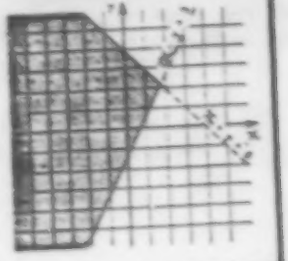
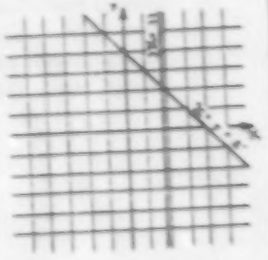
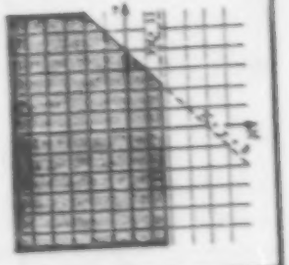
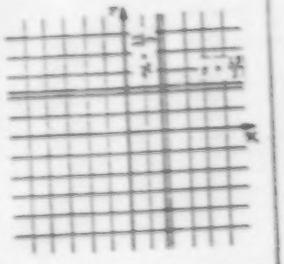
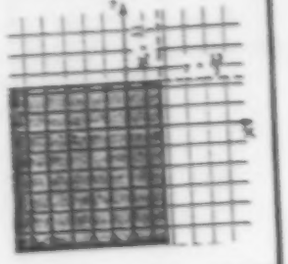
Systems of Inequations	Graphic Representations of the Systems	Graphic Representations of the Solution Sets
$x + y \leq 2$ $2x - y > 3$		
$x + y \geq 2$ $x + y \leq -2$		
$x + y \leq 2$ $2x + 2y \geq -4$		

- The solution set of a system of inequations is made up of ordered [illegible] type (x, y):
  - In the first case : $x \in \mathbb{R}$ and $y \in]-\infty, 0]$;
  - In the second case : $x \in \{\}$ and $y \in \{\}$;
  - In the third case: $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.3 TO GRAPH the solution set of a system of inequations of the first degree in two variables (cont'd).	IV	- There is a clear distinction between the graph of a system and that of its solution set. The pupil must have a firm grasp of this difference if he is to understand the role played by constraints in the construction of the polygon of constraints (Objective 1.5.7).
1.5.4 TO IDENTIFY inequations equivalent to a given inequation of the first degree in two variables.	IV	- Such equivalent inequations are found by means of elementary transformations based on the properties of operations and on the properties of the relationship of order: . Adding a given quantity to each member; . Subtracting a given quantity from each member; . Multiplying or dividing each member by the same non-zero quantity. - Consider the inequation $2x + 3y \leq 9$. When the properties mentioned above are applied we obtain the following equivalent inequations: $2x + 3y - 9 \leq 0$ $4x + 6y - 18 \leq 0$ $-2x - 6y + 9 \geq 0$ etc. - Point out to the pupil that equivalent inequations have the same solution set and the same graph. - ENRICHMENT: The teacher might want to pursue this subject further by studying the equivalence between systems of inequations. For example, is the system $x + y \leq 8$, $2x - y \leq 3$ equivalent to $x \leq 11$, $x - 2y \leq -5$?

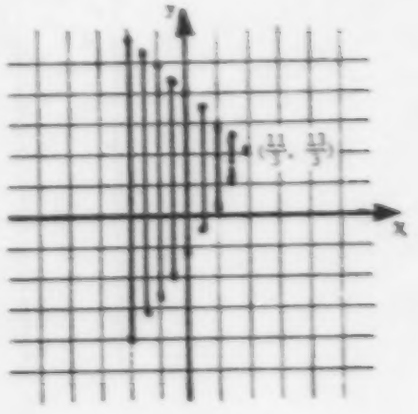
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.5 TO SOLVE the system of equations associated with a system of inequations of the first degree in two variables.	IV	- We have the system of inequations $x + y \leq 8$ $2x - y \leq 3$. The system of equations associated with it is: $x + y = 8$ $2x - y = 3$. - Finding the algebraic solution of the system of equations means "finding the simplest equivalent system." - The teacher will be able to verify, with the aid of the following graphic representations, that looking for the simplest equivalent system of equations brings about a rotation of the lines associated with the initial system, so that these lines become parallel to the axes. The center of rotation corresponds to the center of the bundle which includes these lines.

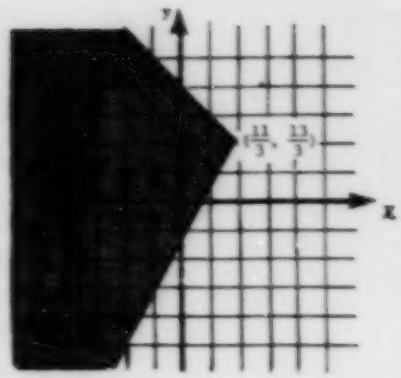
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.5 TO SOLVE the system of equations associated with a system of inequations of the first degree in two variables (cont'd).	IV	

Systems of Inequations	Graphic Representations of the Systems	Graphic Representations of the Solution Sets
$x + y \leq 8$ $2x - y \leq 3$		
$x + y \leq 8$ $3x \leq 11$		
$x \leq \frac{11}{3}$ $y \leq \frac{13}{3}$		

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.5 TO SOLVE the system of equations associated with a system of inequations of the first degree in two variables (cont'd).	IV	- If we consider the systems of inequations associated with these systems of equations, we observe, upon examining their graphs, that they are no longer equivalent. Indeed, the region corresponding to the solution of each system differs, although the point of intersection remains the same. It becomes obvious, therefore, that the algebraic method used for solving systems of equations is not a valid method for solving systems of inequations. For example, the ordered pair (3, 0,5) satisfies the systems of inequation ③ and ②, although it does not satisfy the system ①. The same observation applies to the ordered pair (0, -4) as well as to many others. - It is interesting to note, however, that each value that verifies the inequation $x \leq -\frac{11}{3}$ belongs in the interval $[2x - 3, 8 - x]$. The following table and accompanying graph bring this characteristic into focus.

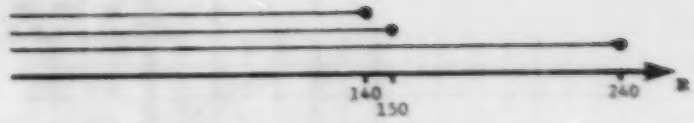
x	$2x-3 \leq y \leq 8-x$
$\frac{11}{3}$	$\frac{13}{3}$
3	$3 \leq y \leq 5$
2	$1 \leq y \leq 6$
1	$-1 \leq y \leq 7$
0	$-3 \leq y \leq 8$
-1	$-5 \leq y \leq 9$
-2	$-7 \leq y \leq 10$

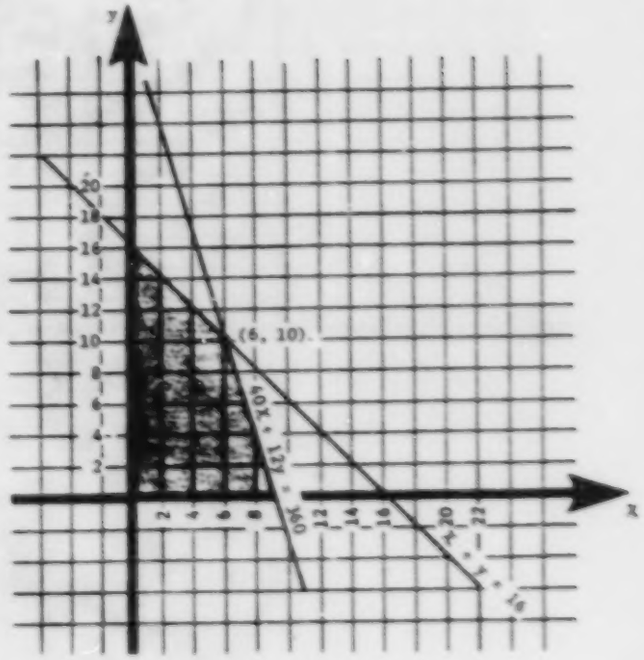


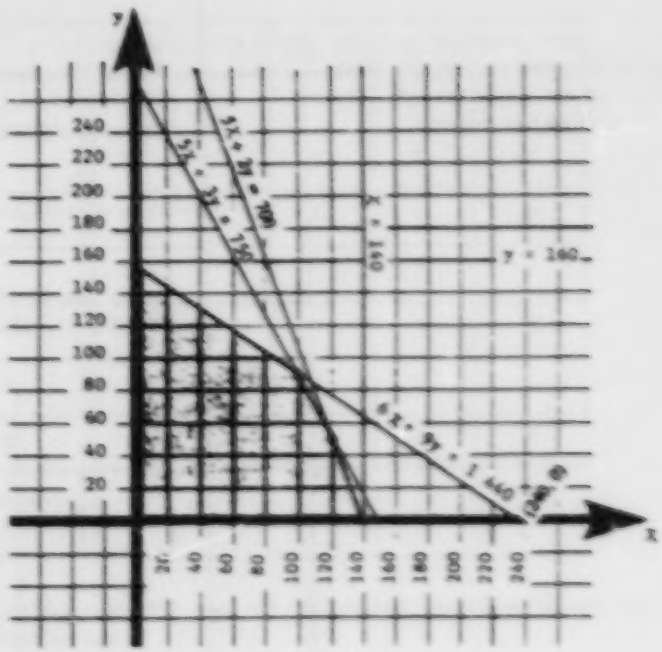
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.5 TO SOLVE the system of equations associated with a system of inequations of the first degree in two variables (cont'd).	IV	$x \leq \frac{11}{3}$ is obtained by adding the two equations of the initial system, member to member, and then by isolating the variable x. The interval $[2x - 3, 8 - x]$ is obtained by transforming $x + y \leq 8$ (1) $2x - y \leq 3$ (2) Indeed, (1) becomes $y \leq 8 - x$ (3) while (2) makes it possible to write $-y \leq 3 - 2x$, hence $y \geq 2x - 3$ (4). (3) and (4) can be written as: $2x - 3 \leq y$ $y \leq 8 - x$. which gives $2x - 3 \leq y \leq 8 - x$ and the interval $[2x - 3, 8 - x]$. - The system of inequation (1) admits as a solution the ordered pairs (x, y), which respect the following constraints: $x \leq \frac{11}{3}$ $2x - 3 \leq y \leq 8 - x$. In this case we obtain the Cartesian graph shown below. This graph is a generalization of the preceding graph which is arrived at by evaluating $2x - 3 \leq y \leq 8 - x$ for all the values of $x \leq \frac{11}{3}$. 

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.5 TO SOLVE the system of equations associated with a system of inequations of the first degree in two variables (cont'd).	IV	- The conclusion is that in the second cycle of secondary school the teacher should confine himself to the use of graphs for solving a system of inequations, even though the solution set of the system of equations associated with it can be found by an algebraic method.
1.5.6 TO EXPRESS by inequations the constraints for a given optimization problem.	V	- This objective can be attained through the following steps: . reading the problem; . determining the variables; . writing the inequations, always respecting the data. - Relevant information on this subject can be found in the notes on General Objective 1, which is about problem solving. - At this point in the mathematization process, the system of inequations thus created constitutes the mathematical model which describes the situation under consideration. - Generally, in the statement of a problem, a sentence or part of a sentence can be associated with a constraint. As the variables are clearly identified it is relatively easy to express this constraint by means of an inequation. It can also be said that generally the function to be optimized is found in the question asked in the problem. This is an example: A teacher is given the task of arranging transportation to a football game for the pupils in his school. He can use two types of buses: big ones and small ones. He may hire at most 16 drivers. If he is given a budget of $360.00 and if rental costs per unit amount to $40.00 for the big buses and $12.00 for the small ones, how many buses of each category should he use if he wants to transport the greatest number of pupils possible, knowing that a big bus has 50 seats and a small one 20?

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.6 TO EXPRESS by inequations the constraints for a given optimization problem (cont'd).	V	- Identifying constraints can sometimes be difficult, either because of the nature of the problem itself or because of the way it is formulated. In the above example, let x be the number of big buses, and y the number of small buses. The number of buses must be greater than or equal to zero: $x \geq 0$ $y \geq 0$. There are at most 16 drivers: $x + y \leq 16$. Transportation costs: $40x + 12y \leq 360$. - The mathematical model which describes the constraints of the problem corresponds to the system of inequations $x \geq 0$ $y \geq 0$ $x + y \leq 16$ $40x + 12y \leq 360$. - Use a broad variety of situations to give the pupil the opportunity to acquire the ability to discover the underlying mathematical models by clearly identifying the constraints inherent in the problems that are analyzed. This is a second example: To collect funds for the graduation ball a group of pupils decides to design and sell souvenir albums. The pupils collect 700 original drawings, 1440 poems, and 750 pages of miscellaneous photographs. A deluxe album includes 5 drawings, 6 poems, and 5 pages of photographs, whereas a regular album includes 2 drawings, 9 poems, and 3 pages of photographs. How many albums of each type can the pupils produce in order to collect the largest amount of money possible if the profit on a deluxe album is $3.00 and the profit on a regular album is $2.00?

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.6 TO EXPRESS by inequations the constraints for a given optimization problem (cont'd).	V	Let x be the number of deluxe albums, and y be the number of regular albums. The number of albums is a natural number greater or equal to zero: $x \geq 0$ $y \geq 0$. The maximum number of albums will be established by: $x \leq \frac{700}{5} = 140$ $x \leq \frac{1440}{6} = 240$ $x \leq \frac{750}{5} = 150$  hence: $x \leq 140$ for the deluxe album. $y \leq \frac{700}{2} = 350$ $y \leq \frac{1440}{9} = 160$ $y \leq \frac{750}{3} = 250$ hence: $y \leq 160$ for the regular albums. - The mathematical model associated with the constraints of this second example is represented by the following system of inequations: $x \geq 0$ $y \geq 0$ $x \leq 140$ $y \leq 160$ $5x + 2y \leq 700$ $6x + 9y \leq 1440$ $5x + 3y \leq 750$

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.7 TO DRAW the polygonal region defined by a system of inequations.	V	- This objective is a generalization of Objective 1.5.2 and requires the mastery of all the skills necessary for its attainment. - The supplementary teaching notes present different ways of constructing the graph of the polygon of constraints. - The system of inequations obtained in the first example given in connection with Objective 1.5.6 makes it possible to obtain the following polygon of constraints: $x \geq 0$ $y \geq 0$ $x + y \leq 16$ $40x + 12y \leq 360$ 

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.7 TO DRAW the polygonal region defined by a system of inequations (cont'd).	V	- The intersection of half-planes (9, 0) always gives a connected polygon of constraints. - The system of inequations associated with the second example (Objective 1.5.6) makes it possible to obtain the following polygon of constraints: $x \geq 0$ $y \geq 0$ $x \leq 140$ $y \leq 160$ $5x + 2y \leq 700$ $6x + 9y \leq 1440$ $5x + 3y \leq 750$  It will be noted here that constraints are not necessary: $x \leq 140$ $y \leq 160$

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.7 TO DRAW the polygonal region defined by a system of inequations (cont'd).	V	The mathematical model can be reduced to the following system: $x \geq 0$ $y \geq 0$ $5x + 2y \leq 700$ $6x + 9y \leq 1440$ $5x + 3y \leq 750$
1.5.8 TO VERIFY if a given point belongs to the polygonal region.	V	- Point out to the pupil that all the points inside the polygon of constraints verify simultaneously all the inequations of the system, whereas a point outside the polygon does not belong to the solution set associated with the system. - This verification can be done by means of substitutions. - Using the first model, we could verify the following ordered pairs: (0, 0), (4, 13), (6, 10), (6, 8), and (8, 6).

	(0, 0)	(4, 13)	(6, 10)	(6, 8)	(8, 6)
$x \geq 0$	T	T	T	T	T
$y \geq 0$	T	T	T	T	T
$x + y \leq 16$	T	F	T	F	F
$40x + 12y \leq 360$	T	T	T	F	F
Belongs to the polygon of constraints	T	F	T	F	F

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.8 TO VERIFY if a given point belongs to the polygonal region (cont'd).	V	The ordered pairs (0, 0) and (6, 10) belong to the polygonal region determined by the system of inequations.

- When we verify the ordered pairs (0, 0), (1, 150), (50, 75), (75, 50) and (150, 50) in the second model, we have:

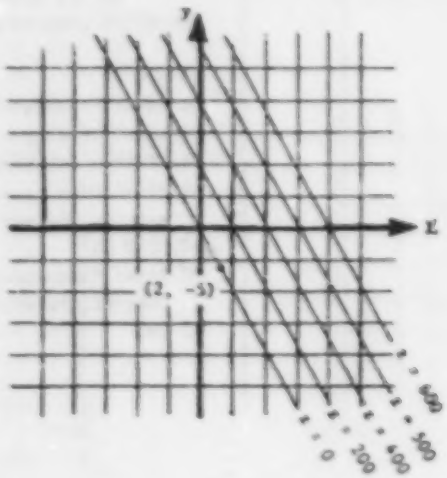
	(0, 0)	(1, 15)	(50, 75)	(75, 50)	(150, 50)
$x \geq 0$	T	T	T	T	T
$y \geq 0$	T	T	T	T	T
$x \leq \frac{350}{3}$	T	T	T	T	F
$y \leq 150$	T	T	T	T	T
$5x + 2y \leq 700$	T	T	T	T	F
$6x + 9y \leq 1440$	T	T	T	T	T
$5x + 3y \leq 750$	T	T	T	T	F
Belongs to the polygon of constraints	T	T	T	T	F

The ordered pairs (0, 0), (1, 150), (50, 75), and (75, 50) belong to the polygonal region determined by the system of inequations.

- The vertices of the polygon of constraints are called the critical points of the polygon. They are found by solving the appropriate systems of inequations.

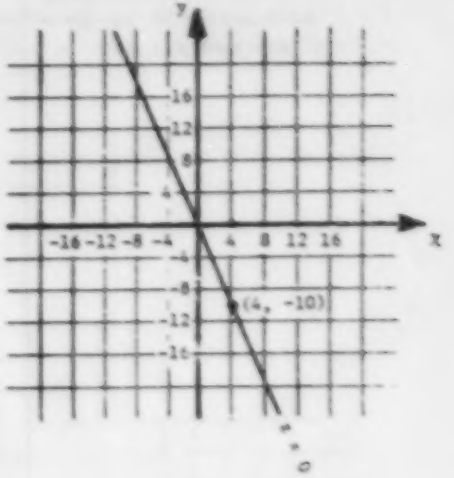
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.8 TO VERIFY If a given point belongs to the polygonal region (cont'd).	V	- In the first example, the critical points are: (0, 0), which originates in the system $x = 0$, $y = 0$ (0, 16), which originates in the system $x = 0$, $x + y = 16$ (6, 10), which originates in the system $x + y = 16$, $40x + 12y = 360$ (9, 0), which originates in the system $y = 0$, $40x + 20y = 360$. - In the second example the critical points are: (0, 0), which originates in the system $x = 0$, $y = 0$ (0, 160), which originates in the system $x = 0$, $6x + 9y = 1440$ (90, 100), which originates in the system $6x + 9y = 1440$, $5x + 3y = 750$ (120, 50), which originates in the system $5x + 3y = 750$, $5x + 2y = 700$ (140, 0), which originates in the system $5x + 2y = 700$, $y = 0$. - The critical points of a polygon of constraints are all solutions of the system of inequations that defines it.

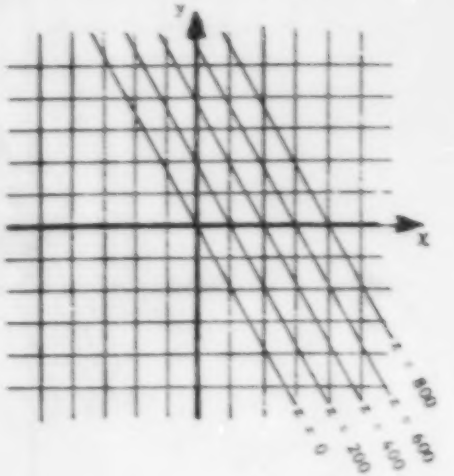
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.8 TO VERIFY if a given point belongs to the polygonal region (cont'd).	V	- It is possible to write a computer program to perform this verification. The BASIC program presented below is associated with the following situation: Lydia buys a rose and a carnation for less than $7.00. If she buys two more roses and sells her carnation, she has to spend at least $3.00. How much must she have paid for one rose and one carnation? The mathematical model for this problem is the following: Let x be the price of a rose, and y be the price of a carnation $x \geq 0$ $y \geq 0$ $x + y \leq 7$ $2x - y \geq 3$. The BASIC program might read like this: 10 CLS 20 INPUT "Indicate the price of a rose."; X 30 INPUT "Indicate the price of a carnation."; Y 40 IF X>=0 AND Y>=0 AND X+Y<=>AND 2*X-Y>=3 THEN 70 50 PRINT "She cannot have paid this price because the selected ordered pair does not satisfy the set of constraints which define the polygon of constraints." 60 GOTO 80 70 PRINT "She may have paid this amount because the selected ordered pair satisfies the constraints which define the polygon of constraints." 80 END

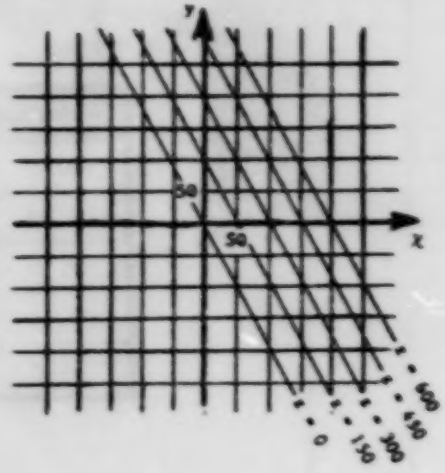
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.9 TO REPRESENT symbolically the function to be optimized.	V	- A problem involving linear programming in $\mathbb{R}^2$ presents a function to be optimized. This function generally is stated by means of two independent variables "x" and "y," of a dependent variable "z" or f(x, y), and of three parameters "A," "B," and "C". This function to be optimized takes the form $z = Ax + By + C$, or $f(x, y) = Ax + By + C$. In the first problem presented under Objective 1.5.6, the question was: "...how many buses of each category should he use if he wants to transport the greatest number of pupils possible, knowing that a big bus has 50 seats and a small one 20?" As we look for the number of vehicles of each type which make it possible to optimize (maximize) the number of pupils, we are able to construct the output function or the function to be optimized. This function can be written as: $z = 50x + 20y$. The graph of this function contains parallel lines whose position in the plane determines the value of z:  - This is how to proceed in order to determine the direction of the function to be optimized as well as a few elements (lines) belonging to this direction.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.9 TO REPRESENT symbolically the function to be optimized (cont'd).	V	1. Let z = 0; z = 50x + 20y becomes 50x + 20y = 0, which allows us to state that the line associated with the equation will go through the origin (Cf. Objective 1.3.1). 2. Transform 50x + 20y = 0; thus 50x + 20y = 0 20y = -50x y = 2,5x 3. Find another ordered pair on this line (Example: (4, -10)). 4. Draw other elements (other lines) of this direction. 5. Find the value of z for each line that is drawn by substituting into z = 50x + 20y the elements of an ordered pair associated with a point located on this line. It is better to select the points at the intersection of these lines with the X-axis:

(x, y)	z
(0, 0)	0
(4, 0)	200
(8, 0)	400
(12, 0)	600
(16, 0)	800

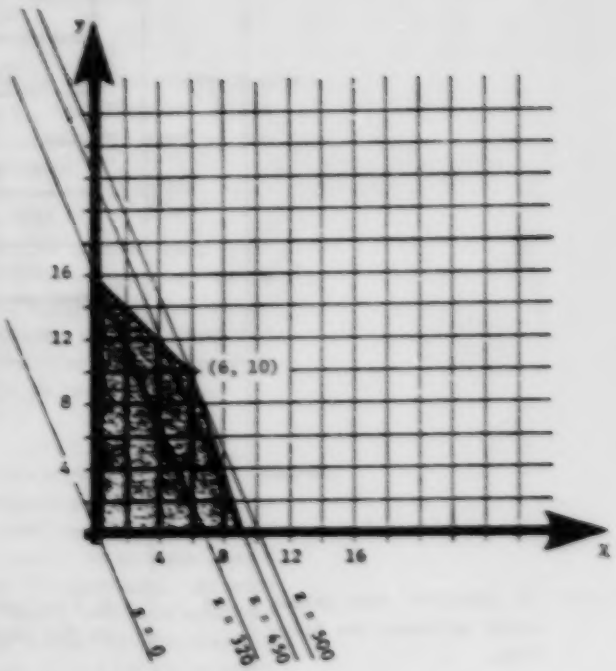




INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.9 TO REPRESENT symbolically the function to be optimized (cont'd).	V	- Once the pupil has discovered the function to be optimized, he should learn how to represent it in the Cartesian plane for a number of given values of z. (Example: For $z = 50x + 20y$, draw the line associated with $z = 90$ in the plane.) It might be beneficial to review with the pupil certain methods and processes presented under Objective 1.3.3. - The question asked in the second problem under Objective 1.5.6 is formulated thus: "How many albums of each type can the pupils produce in order to collect the largest amount of money possible if the profit on a deluxe album is $3.00 and the profit on a regular album is $2.00?" - While looking for the largest amount of money that can be collected, we arrive at the output function or the function to be optimized, which is: $z = 3x + 2y$. The graph of this function contains parallel lines whose position determines the value of z: 

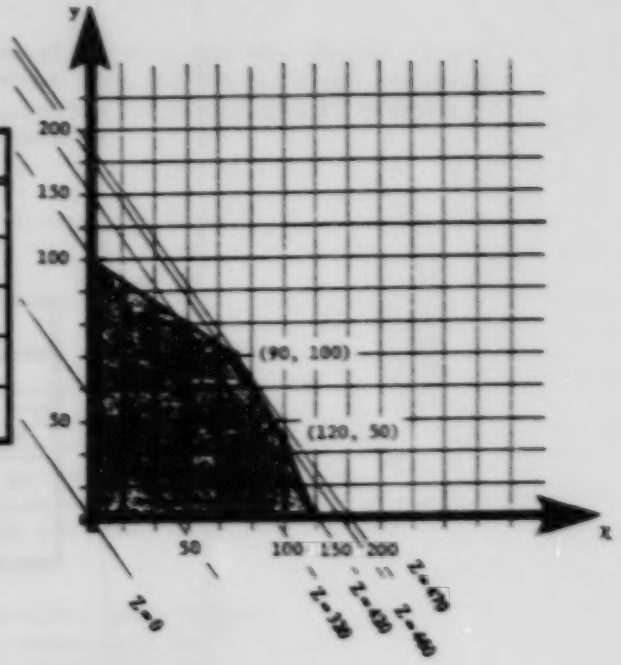
INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.10 TO EVALUATE the function to be optimized at one vertex of the convex polygon.	V	- Evaluating the function to be optimized means finding the value of z for a specific value given to x and for another value given to y. For example, if $z = 50x + 20y$ while $x = 5$ and $y = 7$, the evaluation of the function to be optimized will be $z = 50 \times 5 + 20 \times 7 = 250 + 140 = 390$. We can also write: $f(5, 7) = 390$. - In the first problem presented under Objective 1.5.6, the expression $z = 50x + 20y$ takes the following values for the values of the vertices of the polygon:

(x, y)	$z = 50x + 20y$
(0, 0)	0
(0, 16)	320
(6, 10)	500
(9, 0)	450

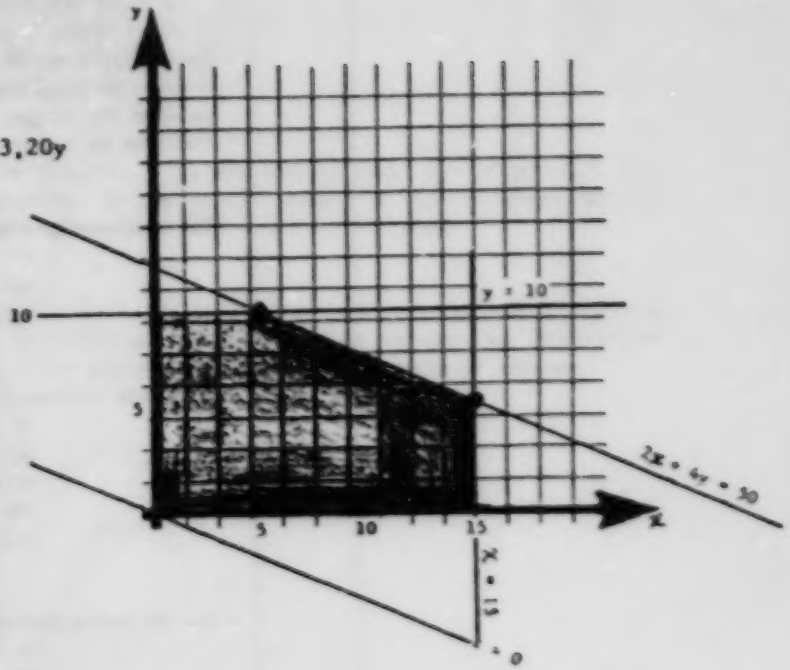


INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.10 TO EVALUATE the function to be optimized at one apex of the convex polygon (cont'd).	V	- In the second problem presented under Objective 1.5.6, the function to be optimized is $z = 3x + 2y$. Its evaluation for the values of the coordinates of the vertices of the polygon will be:
1.5.11 TO IDENTIFY the values which optimize the function.	V	- A function is optimal when its evaluation gives a maximum value or a minimum value, as the case may be.

(x, y)	z = 3x + 2y
(0, 0)	0
(0, 160)	320
(90, 100)	470
(120, 50)	460
(140, 0)	420



INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.11 TO IDENTIFY the values which optimize the function. (cont'd)	V	- In the first table (Objective 1.5.10) we observe that z is maximum when $X = 6$ and $y = 10$. This means that in the first example (Objective 1.5.6), the teacher should choose 6 big buses and 10 small ones if he wants to transport the greatest number of pupils possible. It is possible to check the accuracy of this result by means of a graph. - Obviously, z could take a countless number of values greater than 500. However, in such a case the line associated with the function to be optimized does not bisect the convex polygon any longer, and the ordered pairs (x, y) that characterize each of the points on the line no longer verify the system of inequations. - The teacher might help the pupil to discover the following theorem: If the optimal value of the function to be optimized exists, it is found at least at one vertex of the convex polygon. - From this theorem it is possible to discover another one: If a line associated with a function to be optimized optimizes that function at two consecutive vertices of a convex polygon, then all the points located on the line segment connecting these vertices optimize the function. - The following contextualized problem illustrates this second theorem: To complete a given project a company must use the services of 15 apprentice-electricians and 10 electricians at the most. In no case should the job take more than 50 man-hours, as each apprentice must work 2 hours and each electrician 4 hours. Find the number of employees in each category the company must hire if it wants to maximize the value of the contract and if it offers $21.60 an hour for an apprentice and $43.20 an hour for an electrician.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.11 TO IDENTIFY the values which optimize the function (cont'd).	V	Here are the mathematical model and graph associated with this situation: Let x be the number of apprentices, and y be the number of electricians. $x \leq 15$ $y \leq 10$ $2x + 4y \leq 50$ $z = 21{,}60x + 43{,}20y$ The evaluation of z at each vertex gives: (x,y) | z (0, 0) | 0 (0, 10) | 432 (5, 10) | 540 (15, 5) | 540 (15, 0) | 324  Two vertices maximize the function, therefore all the points located on the line segment connecting these two vertices also maximize the function.

INTERMEDIATE OBJECTIVES	LEVELS	NOTES
1.5.11 TO IDENTIFY the values which optimize the function (cont'd).	V	In order to satisfy the constraints of this contract and to maximize its returns, the company might choose one of the following solutions: 5 apprentices and 10 electricians, 7 apprentices and 9 electricians, 9 apprentices and 8 electricians, 11 apprentices and 7 electricians, 13 apprentices and 6 electricians, 15 apprentices and 5 electricians. - In the second table (Objective 1.5.10), z is the maximum when $x = 90$ and $y = 100$. Profits will be maximum, therefore, if the pupils produce 90 deluxe albums and 100 regular albums. We can check the accuracy of this result on the graph. - The computer can be used to find optimal solutions. Writing a computer program for this purpose will enable the pupil to synthesize a large number of terminal and intermediate objectives presented under the first general objective of this program of studies.

SUPPLEMENTARY TEACHING NOTES

(A) Numbers

Numbers are powerful tools which Man uses in his efforts to unravel the mysteries of the universe. Numbers carry with them their own aura of mystery. It is said, for example, that 7 is a lucky number, whereas it is a popular belief that 13 is unlucky. Many public buildings do not have a thirteenth floor. Some numbers are known as perfect numbers, others as triangular numbers, square numbers, polygonal numbers, and so on. And there are magic squares and number sequences with intriguing patterns.

Some knowledge of the history of numbers might enhance the pupil's understanding when he begins his study of the number system.

Archeological research has shown that prehistoric society had an articulated language that included a numeration system. Counting was done by means of descriptive symbols, possibly based on the principle of one-to-one correspondence (a pile of pebbles for a number of sheep, for example, or marks on bones, wood, rocks, and other materials). None of this, however, means that the concept of numbers <u>per se</u> existed in prehistoric society. Finger counting led to groupings by twos, threes, or fives.

The Romans are mistakenly believed to have used the symbols I, V, X, L, C, D, M, D, M, etc. In fact, this modern way of writing Roman numerals goes back only to the Middle Ages. The Romans rarely used large numbers and generally did not use the horizontal mark to indicate large numbers. This mark served to differentiate words from numbers. The subtraction principle, which involves taking away at most one unit from a basic symbol, was not always respected. For example, the Romans wrote IIX instead of VIII or IIXX instead of XVIII.

The numeration system used by the Romans seems to follow the following rules:

1. Any basic symbol placed on the right of a symbol of equal or higher value must be added to it.

   Example: VI = 6; MLVI = 1056.

2. Any basic symbol placed on the left of a symbol of higher value must be subtracted from it.

   Example: IV = 4; XL = 40.

These rules, however, do not always apply and they can be difficult to interpret. Take the number XIV, for example. If we follow the first rule, we would add I to X, whereas if we applied the second rule we would subtract I from V. We know, however, that XIV represents the number 14.

Here is the ancient version of the Roman numerals:

I, V, Λ, ⅄, X, ⅃, ⅂, L, ⊕, ⊕, C, O, Θ, Λ, ⊃, D, C⊃, ⅄, M, ⊕ et C⊃

Here is the modern equivalent:

I, V, X, L, C, D, M, $\overline{D}$, and $\overline{M}$.

The Chinese, like the Greeks, used two notation systems:

(a) A system based on the multiplicative principle

一	-	1
二	-	2
三	-	3
四	-	4
五	-	5
六	-	6
七	-	7
八	-	8
九	-	9
十	-	10
百	-	100
千	-	1000

Numerical symbols in odd positions from left to right or from bottom to top are multiplied by the numeral that comes next.

Thus, 7534 is represented by:

7x1000 5x100 3x10 4

七千 五百 三十 四

(b) A system based on the principle of place value

vertical	𝍠	𝍡	𝍢	𝍣	𝍤				
	1	2	3	4	5				
	𝍥	𝍦	𝍧	𝍨					
	6	7	8	9					
horizontal	𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱
	1	2	3	4	5	6	7	8	9

When these two types of symbols are used alternately, starting with the units (the digit on the right) which by convention are vertical, it is possible to represent any number of any magnitude.

The Chinese used the abacus as a quick and effective computing tool.

Here is how the written forms of our numerals have evolved historically:

TODAY	1	2	3	4	5	6	7	8	9	0
400 YEARS AGO	١	٢	٣	٤	٥	٦	٧	٨	٩	٠
1000 YEARS AGO	١	٢	٣	٤	٥	٦	٧	٨	٩	٠
1500 YEARS AGO	١	٢	٣	٤	٥	٦	٧	٨	٩	٠
2200 YEARS AGO	١	٢	٣	٤	٥	٦	٧	٨	٩	

Different types of groupings are in use in various parts of the world today. The following table shows the frequency (in percentages) with which different bases are used. These bases were in use in America at the turn of the century.

Bases	10	5 or 5 and 10	20 or 5 and 20	4	3, 8, 12, 60
% of use	49	33	10	4	4

These systems are based on three methods of grouping:

- grouping by adding unit to unit,
- grouping based on the principle of repetition (additive positional system),
- grouping based on the principle of place value.

Today, with the widespread use of electronic machines, the base-two numeration system is becoming increasingly important. Base two is particularly effective for studying logical, electrical, and other circuits. The base-two system also makes the construction of mathematical models for dichotomous situations easy. Boolean algebra provides the ideal structure for such an analysis of real situations.

Regardless of the base, numbers can be grouped into sets depending on whether or not certain operations can be performed on these sets. As operations are essential in the performance of any task involving numbers, the teacher must have a full understanding of their importance and scope.

> Consider a set A. f is an operation on A if and only if A x A can be applied to A so that any pair of A x A that is so applied yields an answer in A.

Symbolically an operation can be defined thus:

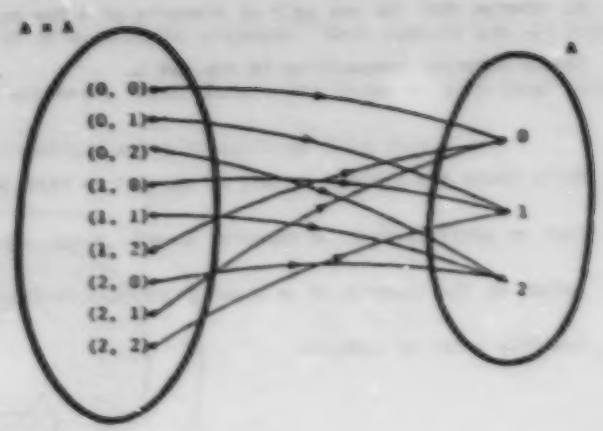
$$\forall x, y, z \in A, f : A \times A \to A : (x, y) \mapsto z = x \top y$$

Verification by means of a finite set of elements, modulo 3 for example, should bring into focus the characteristics of this definition for addition operations.

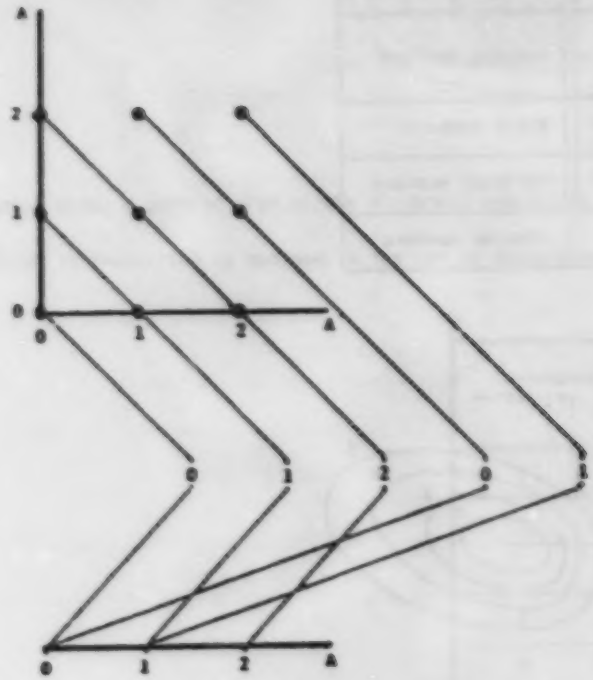
Let $A = \{0, 1, 2\}$

$\bullet : A \times A \to A : (x, y) \mapsto x + y \text{ (modulo 3)}$

We can then construct the arrow graph on the right:



This network graph illustrates the same situation:



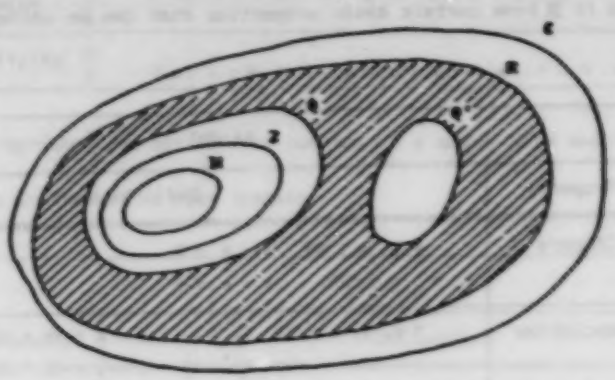
We observe that for any pair of elements of A the answer is found in A. This is an illustration of an operation on the set A or of the law of internal composition in the set A.

While there are a broad variety of operations that can be performed on numbers, addition is the basic operation. Multiplication is in fact an addition that is repeated several times, while subtraction is the addition of the opposite of a number. Division is multiplication by the inverse of a number, and exponentiation is repeated multiplication. Each of these operations makes it possible to identify a set of numbers:

Operations		Sets of Numbers
Addition	+	Natural numbers
Multiplication	x	
Subtraction	-	Whole numbers
Division	/	Rational numbers
Exponentiation	†	Complex numbers

The Venn diagram below makes it possible to visualize this situation:

It is also possible to identify decimal numbers whose decimal part is made up of a periodic sequence. Such numbers are all rational numbers. With other numbers, the decimal part is made up of a non-periodic sequence. Such numbers are known as irrational numbers. The union of these two sets of numbers gives the set of real numbers. The following diagram illustrates these facts:



The table below gives a general view of the different operations in relation to the sets of numbers. The tick mark indicates that the operation which is identified is defined in the set of numbers considered:

Operations	Sets of Numbers					
	ℕ	ℤ	ℚ	ℚ'	ℝ	ℂ
+	✓	✓	✓		✓	✓
×	✓	✓	✓		✓	✓
−		✓	✓		✓	✓
/			✓		✓	✓
√						✓

It should be noted that the exponentiation operation (↑) is defined only in the set of complex numbers ($\mathbb{C}$). It is possible, however, to perform operations on a subset of $\mathbb{C}$. We have, for example, the following operations: $3 \uparrow 2 = 3^2 = 9$ $2 \uparrow \frac{1}{2} = 2^{\frac{1}{2}} = \sqrt{2}$. Although subtraction is not defined in $\mathbb{N}$, the elementary school pupil performs subtractions in the set of natural numbers. Addition and multiplication operations in $\mathbb{R}$ have certain basic properties that can be called axioms. These axioms are identified and defined in the following table:

AXIOMS	
Property	**Definition**
Closure	$\forall a, b \in \mathbb{R}$: $a + b \in \mathbb{R}$ $a \cdot b \in \mathbb{R}$
Associative	$\forall a, b, c \in \mathbb{R}$: $(a + b) + c = a + (b + c)$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Cancellation of Addition	$\forall a \in \mathbb{R}, \exists b \in \mathbb{R}$: $a + b = a \Rightarrow b = 0$ $\forall a \in \mathbb{R}, \exists b \in \mathbb{R}$: $a \cdot b = a \Rightarrow b = 1$
Symmetry	$\forall a \in \mathbb{R}, \exists -a \in \mathbb{R}$: $a + (-a) = 0 = (-a) + a$ $\forall a \in \mathbb{R}^*, \exists a^{-1} \in \mathbb{R}$: $a \cdot a^{-1} = 1 = a^{-1} \cdot a$
Distributive	$\forall a, b, c \in \mathbb{R}$: $a \cdot (b + c) = a \cdot b + a \cdot c$ $(b + c) \cdot a = a \cdot b + a \cdot c$
Commutative	$\forall a, b \in \mathbb{R}$: $a + b = b + a$ $a \cdot b = b \cdot a$

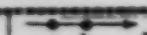
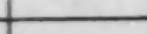
These properties can be applied for the purpose of evaluating complex numerical expressions. These properties also make it possible to solve equations by successively transforming them into equivalent expressions. They are then used in conjunction with the properties of additive inverse and simplifying.

AXIOMS	
Property	Definition
Additive inverse	$\forall a, b, c \in \mathbb{R}: a = b \Leftrightarrow a + c = b + c$
Simplifying	$\forall a, b \in \mathbb{R}, \forall c \in \mathbb{R}^*: a = b \Leftrightarrow a \cdot c = b \cdot c$

The teacher may find it helpful to represent the set of real numbers on a number line and to order each element of the set. The table below illustrates order relationships in the set of real numbers.

ORDER	
Relationship	Definition
Is less than	$\forall a, b \in \mathbb{R}, \forall c \in \mathbb{R}_0^+: a + c = b \Leftrightarrow a < b$
Is less than or equal to	$\forall a, b \in \mathbb{R}, \forall c \in \mathbb{R}_0^+: a + c = b \Leftrightarrow a \leq b$
Is greater than	$\forall a, b \in \mathbb{R}: a > b \Leftrightarrow b < a$
Is greater than or equal to	$\forall a, b \in \mathbb{R}: a \geq b \Leftrightarrow b \leq a$

These definitions of order relationships make it possible to determine intervals in the set of real numbers.

Interval	Symbol		Definition
Closed interval	$[a, b]$		$[\forall x \in \mathbb{R} \mid a \leq \forall x \leq b]$
Open interval	$]a, b[$		$[\forall x \in \mathbb{R} \mid a < \forall x < b]$
Half-open interval on the right	$[a, b[$		$[\forall x \in \mathbb{R} \mid a \leq \forall x < b]$
Half-open interval on the left	$]a, b]$		$[\forall x \in \mathbb{R} \mid a < \forall x \leq b]$

Order relationships in the set of real numbers can be used in conjunction with the axioms pertaining to real numbers and the axioms of order presented below for the purpose of transforming inequations into equivalent inequations.

AXIOM	
Property	Definition
Additive inverse	$\forall a, b, c \in \mathbb{R}: \; a < b \Leftrightarrow a + c < b + c$
Simplifying	$\forall a, b \in \mathbb{R}, \forall c \in \mathbb{R}_0^+: \; a < b \Leftrightarrow a \cdot c < b \cdot c$
	$\forall a, b \in \mathbb{R}, \forall c \in \mathbb{R}_0^-: \; a < b \Leftrightarrow a \cdot c > b \cdot c$

Finally, there are a number of theorems pertaining to exponentiation operations. These theorems and the axioms, concepts, and facts that were discussed earlier provide the basis for a numerical algebra that is sufficiently effective for mathematizing situations and for performing operations on the models. Here are the theorems that apply to exponentiation operations:

THEOREMS
$\forall a \in \mathbb{R}: a \div 0 = a^0 = 1$
$\forall a \in \mathbb{R}^*: a \div -1 = a^{-1} = \frac{1}{a}$
$\forall a \in \mathbb{R}, \forall m, n \in \mathbb{Q}: a \div m \times a \div n = a^m \times a^n = a^{m+n}$
$\forall a \in \mathbb{R}^*, \forall m, n \in \mathbb{Q}: \frac{a \div m}{a \div n} = \frac{a^m}{a^n} = a^{m-n}$
$\forall a \in \mathbb{R}, \forall b \in \mathbb{R}^*, \forall n \in \mathbb{Q}: \left(\frac{a}{b}\right) \div n = \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
$\forall a \in \mathbb{R}, \forall m, n \in \mathbb{Q}: (a \div m) \div n = (a^m)^n = a^{m \times n}$
$\forall a \in \mathbb{R}, \forall n, n \in \mathbb{Z}: a \div \frac{n}{n} = a^{n/n} = \sqrt[n]{a^n}$

(B) Matrices

Situations can often be translated by networks or graphs. It is then relatively easy to transform these networks and graphs into matrices. We only have to count the number of paths linking two nodes of the network. Thus,

gives this matrix

	A	B	C	D
A	1	1	0	0
B	1	0	3	0
C	0	3	0	1
D	0	0	1	0

and inversely, this matrix

	A	B	C
A	2	1	0
B	1	1	2
C	0	2	0

corresponds to this network

Conversion into matrices is also feasible with networks in which the paths have direction arrows.

the corresponding matrix is

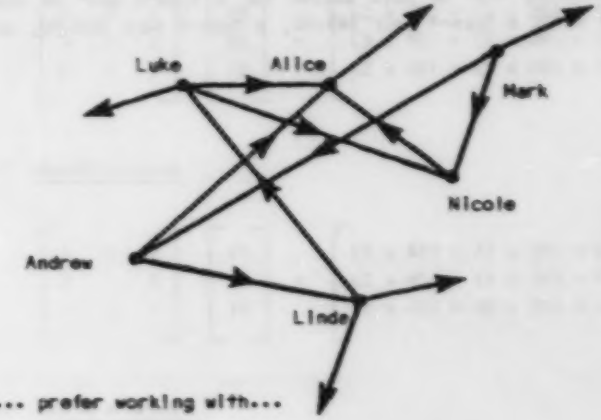
	A	B	C	D
A	0	1	0	0
B	0	1	1	1
C	0	0	0	0
D	1	0	0	0

The ability to mathematize is also developed through such activities as recording data, constructing data tables, and storing information with the aid of matrices. For example, a milkman can very well record his week's sales this way:

$$\begin{bmatrix} 9 & 3 & 0 & 5 & 3 & 8 \\ 6 & 0 & 0 & 3 & 0 & 6 \\ 6 & 6 & 0 & 1 & 3 & 6 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Each column designates a day of the week (M, T, W, T, F, S, S), and each line designates the position of each customer on his route.

The ability to mathematize comes into play when constructing a sociogram, for determining one's favorite friends, for example.



... prefer working with...

	Luke	Alice	Andrew	Mark	Nicole	...
Luke	0	1	0	0	0	...
Alice	0	0	1	0	1	...
Andrew	0	1	0	1	0	...
Mark	0	0	1	0	1	...
Nicole	1	1	0	1	0	...
.	.	.	.	.	.	...
.	.	.	.	.	.	...

Writing matrices, however, does not necessarily help develop mathematical skills, unless matrices are treated mathematically and the results are interpreted.

Example:

At the end of September a shoe store owner has in stock

		Types A	B	C
Sizes	20	4	2	3
	22	1	3	5
	24	3	1	5

In October he buys

		Types A	B	C
Sizes	20	1	3	1
	22	4	0	1
	24	0	2	0

During the month he sells

		Types A	B	C
Sizes	20	3	2	2
	22	2	1	1
	24	1	0	1

What is his inventory at the end of October? How much is his stock worth, knowing that he pays $32.00 for a type-A pair of shoes, $20.00 for a type-B pair, and $12.00 for a type-C pair, and knowing that he sells a type-A pair $45.00, a type-B pair $29.00, and a type-C pair $19.00? What is the amount of profit he makes?

Solutions

a) **Inventory**

$$\begin{bmatrix} 4 & 2 & 3 \\ 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 4 \\ 5 & 4 & 3 \\ 3 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5 & 4 \\ 5 & 4 & 3 \\ 3 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 3 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

b) **Stock value**

$$\begin{bmatrix} 2 & 3 & 2 \\ 3 & 3 & 2 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 45 \\ 29 \\ 19 \end{bmatrix} = \begin{bmatrix} (2 \times 45) + (3 \times 29) + (2 \times 19) \\ (3 \times 45) + (3 \times 29) + (2 \times 19) \\ (2 \times 45) + (3 \times 29) + (4 \times 19) \end{bmatrix} = \begin{bmatrix} 215 \\ 260 \\ 253 \end{bmatrix} = \$728.00$$

c) **Month's sales**

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 45 \\ 29 \\ 19 \end{bmatrix} = \begin{bmatrix} (3 \times 45) + (2 \times 29) + (2 \times 19) \\ (2 \times 45) + (1 \times 29) + (1 \times 19) \\ (1 \times 45) + (0 \times 29) + (1 \times 19) \end{bmatrix} = \begin{bmatrix} 212 \\ 138 \\ 64 \end{bmatrix} = \$414.00$$

d) **Month's orders**

$$\begin{bmatrix} 1 & 3 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 32 \\ 20 \\ 12 \end{bmatrix} = \begin{bmatrix} (1 \times 45) + (3 \times 29) + (1 \times 19) \\ (4 \times 45) + (0 \times 29) + (1 \times 19) \\ (0 \times 45) + (2 \times 29) + (0 \times 19) \end{bmatrix} = \begin{bmatrix} 104 \\ 140 \\ 40 \end{bmatrix} = \$284.00$$

e) Profits

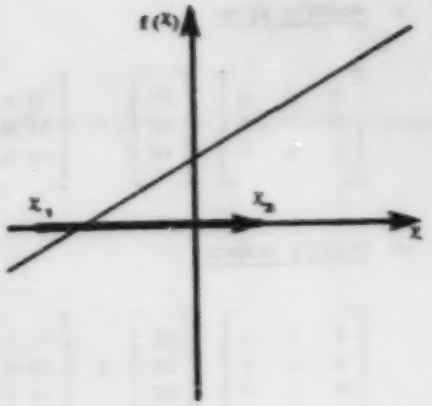
$414.00 - $284.00 = $130.00

When the study of matrices has been completed, the teacher might consider teaching the pupils how to use them in geometry, as the algebraic formation of matrices is relevant to the study of plane transformations. A knowledge of reflections, translations, rotations, analytical geometry and trignometry, is useful for a more in-depth exploration of matrices and for the construction of the matrices of Euclidian isometries. This subject is explored in connection with the transformations presented in General Objective 2.

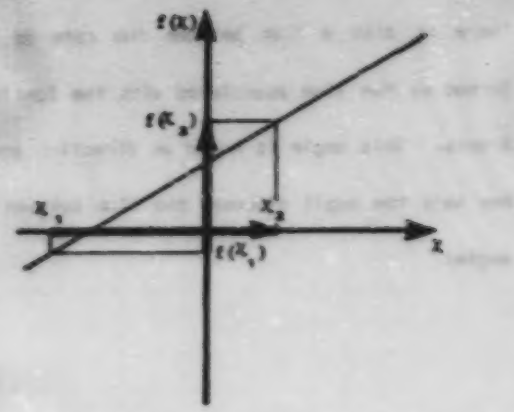
(C) Affine Functions

The study of the slope of an affine function of the form $f(x) = mx + b$ has a variety of applications, especially if it is approached from the point of view of variation.

In the graph of an affine function reproduced on the right, it is possible to cause a positive variation in the value of "x" by going from x_1 to x_2. This variation can be expressed by Δx, that is $x_2 - x_1$.



Such a variation in the values attributed to "x" causes the values attributed to "y" to vary. The graph on the right illustrates this variation, which can be expressed by $\Delta f(x)$ or $f(x_2) - f(x_1)$. The rate of change is found by calculating the quotient



$$\frac{\Delta f(x)}{\Delta x}$$

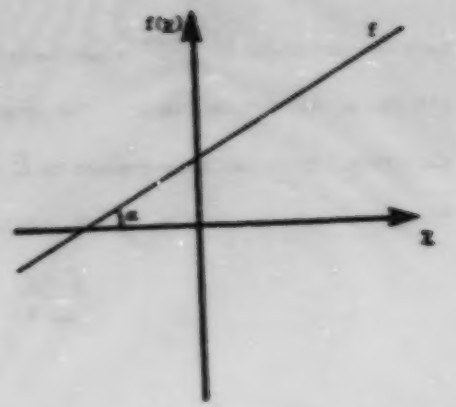
which is equivalent to:

$$\frac{f(x_2) - f(x_1),}{x_2 - x_1}$$

or to

$$\frac{y_2 - y_1}{x_2 - x_1} = m.$$

The slope of the function is given by this rate. The slope is sometimes called "direct-proportionality ratio." The rate of change is constant and exclusive for a given function.



There is also a link between the rate of change (slope) and the angle formed by the line associated with the function $f(x) = a\,x + b$ and by the X-axis. This angle is known as direction angle. The following activities may help the pupil discover the link between rate of change and direction angle:

1. Ask the pupil to construct the Cartesian graph of a given affine function very accurately.

2. Ask the pupil to evaluate the rate of change of that function based on its Cartesian graph.

3. Ask the pupil to use a protractor to measure to the nearest degree the direction angle associated with that function.

4. Ask the pupil to use a pocket calculator to find the value of the tangent of the direction angle. (Check whether the calculator is set on "degree" or on "radian").

5. Ask the pupil to compare the result obtained by means of the calculator with the evaluation of the rate of change.

The pupil should reach the conclusion that:

$$\tan \alpha \approx \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \tan \alpha \text{ for } m.$$

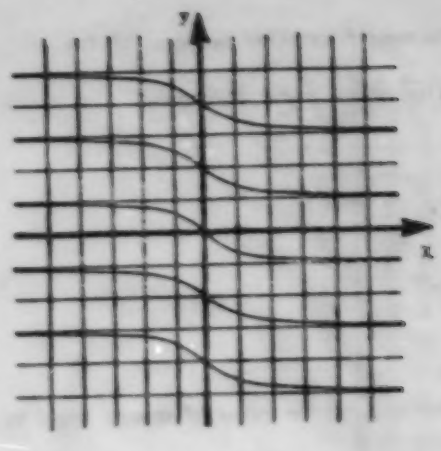
If the teacher makes the pupils evaluate the mean of the rates of change found by the whole class, as well as the mean of the values of $\tan \alpha$, it is possible to conclude that:

$$\tan \alpha \approx m.$$

Based on the study of the Cartesian graph of a number of affine functions of the form $f(x) = mx + b$, it can be observed that if:

- $\alpha = 0°$, then $m = 0$;
- α approximates 90°, then m approximates an infinite value;
- $\alpha = 90°$, then m is not defined (on the calculator this is quite obvious);
- $\alpha > 90°$ and α approximates 180°, then m approximates 0 but negatively;
- $\alpha = 180°$, then $m = 0$.

The Cartesian graph of all ordered pairs of the form (α, m) provides the representation of $\tan \alpha$.



As an illustration of the concept of affine function and of its usefulness, let us consider this example. If a motorist drives from Quebec City to Montreal at a constant speed of 80 km/h, what distance does he/she cover in one hour, in 2,5 hours, and in 3 hours?

After examining the statement of the problem it is possible to construct the following mathematical model:

distance travelled = speed x time (1)

As the speed is constant, the distance is a function of time. Here we have an example of the direct variation function of the form $f(x) = mx$. If s stands for the speed and t represents the time, then

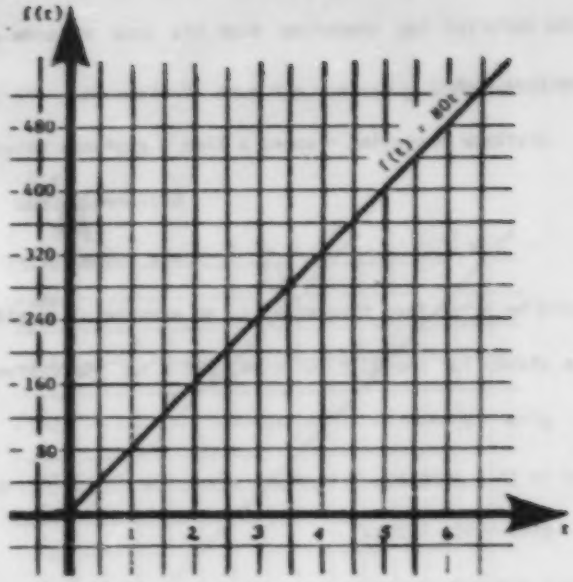
$$f(t) = s\,t \qquad (2)$$

It is advisable to use as few symbols as possible and to write simply in order to make the situation easier to understand. Thus, although equation (1) and equation (2) mean the same thing, the former is more difficult to handle mathematically.

The problem can be mathematized in this way:

$$f(t) = 80\,t \qquad (3)$$

The graph of this model is shown on the right.



Operations on this model give the following results:

$f(1) = 80 \times 1$
$= 80$
$f(2,5) = 80 \times 2,5$
$= 200$
$f(3) = 80 \times 3$
$= 240$

These results can be interpreted thus:

(a) In 1 hour the motorist travels 80 km;

(b) In 2,5 hours, the motorist travels 200 km;

(c) In 3 hours, the motorist goes a distance of 240 km;

Let us now calculate the amount of time it takes someone to drive from Québec City to Montreal if we know that the distance between the two cities is 267 km.

Equation (2) then becomes:

$$267 = 80t$$
$$\frac{267}{80} = t$$
$$t = 3,3375.$$

We may conclude, therefore, that the motorist needs 3,3375 hours to travel 267 km at a speed of 80 km/h.

Let us say that the motorist lives in Rivière-du-Loup, a town located 206 km east of Québec City. If we want to determine the distance that the motorist has travelled from his home at some point during the trip between Montreal and Québec City, we might use this mathematical model:

distance travelled = speed x time + distance between Rivière-du-Loup and Québec City. (4)

It should be noted that this model is an example of an affine function: $f(x) = mx + b$.

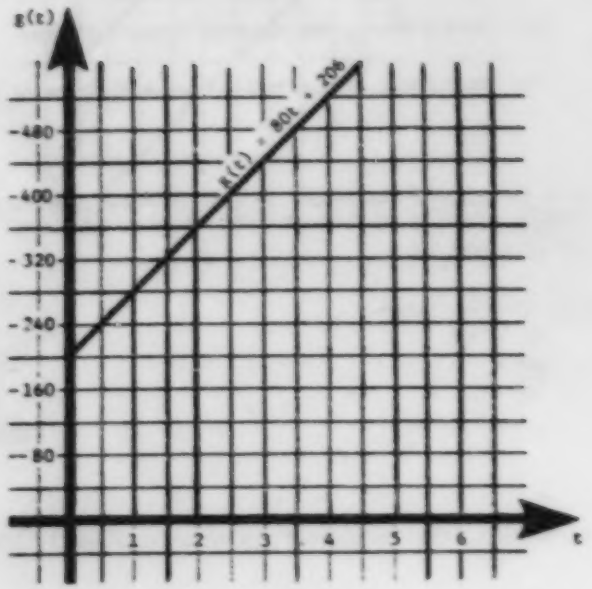
Here, s stands for speed, t for time, and d for the distance between Rivière-du-Loup and Québec City. We have then this equation:

$$g(t) = st + d \qquad (5)$$

Applied to this problem, this model gives the following equation:

$$g(t) = 80t + 206.$$

The graph presented on the right illustrates the situation.



Then, g (1), g (2,5), and g (3) can be evaluated thus:

g (1) = 80 x 1 + 206
= 286

g (2,5) = 80 x 2,5 + 206
= 406

g (3) = 80 x 3 + 206
= 446

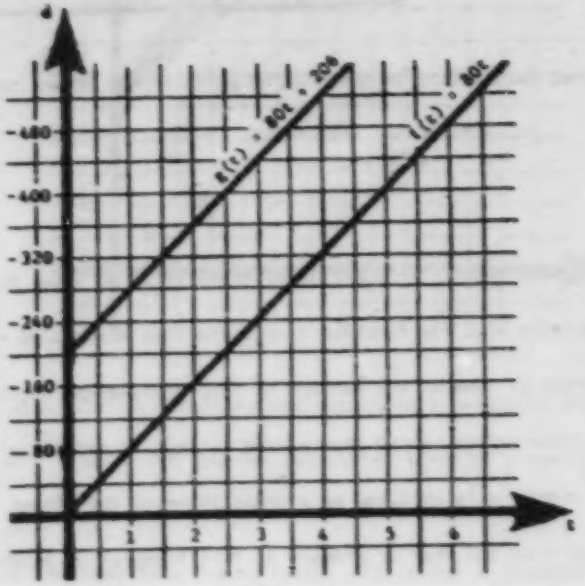
Consequently:

(a) After 1 hour the motorist is 286 km from home;
(b) After 2,5 hours he/she is 406 km from home;
(c) After 3 hours he/she is 446 km from home;

Similarly, g (o) = 206 means that before his/her departure, the motorist is 206 km from home.

By superimposing the Cartesian graphs of models (3) and (6) onto each other it is possible to make the following observations:

1. The two lines are parallel.
2. The rate of change is identical for the two functions.
3. The rate of change here corresponds to the speed.



It can be concluded that the graph of affine functions with the same rate of change is made up of parallel lines. Indeed,

$f(t) = 80t$ (1)

$g(t) = 80t + 206$ (2)

By substituting (1) in (2) we have:

$g(t) = f(t) + 206$ (3)

If we examine (3), two things become evident:

1. The rate of change is identical for the functions f and g.
2. The ordered pairs of the function g are found by adding 206 to the second element of the ordered pairs of f. Indeed:

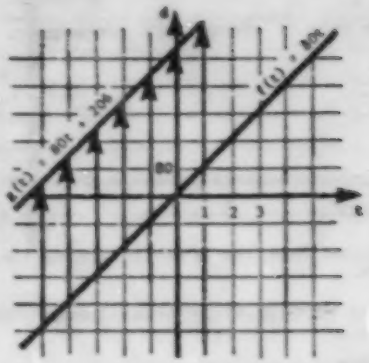
t	f(t)	f(t) + 206	g(t)
0	0	206	206
1	80	286	286
2,5	200	406	406
3	240	446	446

and

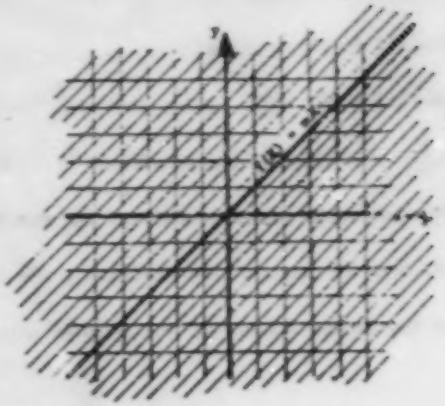
t	(t, f(t))	(t, g(t))
0	(0, 0)	(0, 206)
1	(1, 80)	(1, 286)
2,5	(2,5, 200)	(2,5, 406)
3	(3, 240)	(3, 446)

A close examination of the ordered pairs in the above tables reveals that:

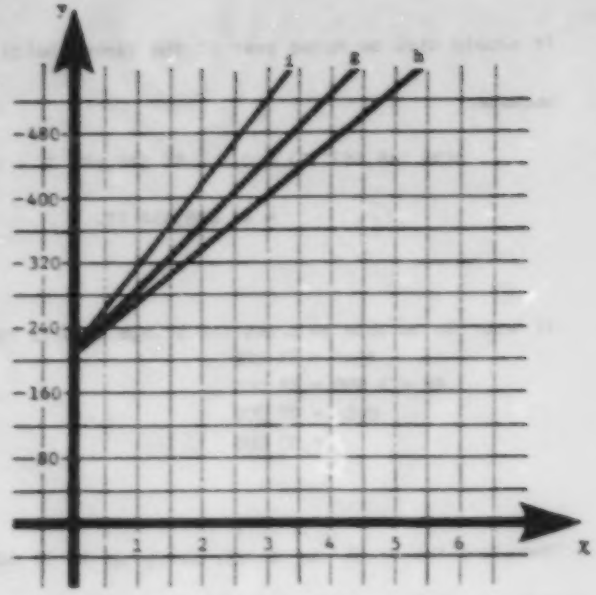
$$(t, g(t)) = (t, f(t) + 206)$$
$$= (t + 0, f(t) + 206)$$
$$= (t, f(t)) + (0, 206)$$



This means that it is possible by translation to slide the line associated with the function f onto the line associated with the function g. One of the vectors of this translation is (0,206). This observation can be generalized so that it can be said that $f(x) = mx + b$ is obtained by a translation of the linear function $f_1(x) = mx$ and that an element of this translation is the pair (0,b).



In a special situation in which "m" is constant and "b" is variable, $y = mx + b$ defines a family of parallel lines known as a direction. Here, $y = 80x + b$ represents this family of lines.



To complete the preceding example let us consider the case of two motorists from the town of Rivière-du-Loup. Both leave from Quebec City heading for Montreal. One drives at a constant speed of 70 km/h and the other at a constant speed of 90 km/h. It is possible to construct the graph of the three functions on the same Cartesian plane:

$g(t) = 80t + 206$
$h(t) = 70t + 206$
$i(t) = 90t + 206$

When we examine this graph we observe that the rate of change is different for each of the three functions, but that the three lines always cross the y-intercept 206.

Here also it is possible to generalize and to conclude that if "m" is variable while "b" is constant, the result is a bundle of lines. This bundle constitutes another type of family of lines. Here, the expression $y = mx + 206$ can be associated with this family of lines.

Here is a second example. An empty tanker truck has a mass of 3800 kg. Its cylindrical tank has a length of 6 m and a diameter of 3 m. The oil it contains has a mass of 0,8 kg per liter (the density of oil is 0,8). If the truck has to cross a bridge whose maximum authorized capacity is 35 000 kg, what is the maximum amount of oil the truck can carry and still be able to cross the bridge safely?

The problem can be mathematized by means of the following model:

$0 \leq n \leq 56\,548{,}667\,76$ — n: number of liters in the tank ($6 \times 3 \times \pi \times 1000 = 56\,548{,}667\,76$)

$f(n) = 0{,}8n + 3800$ — f(n): the mass of the truck for n litres of oil;

$f(n) \leq 35\,000$ — maximum load allowed

It should also be noted that if the tanker holds 56 548,667 76 litres, the total mass of the truck is evaluated at 49 038,934 21 kg, because:

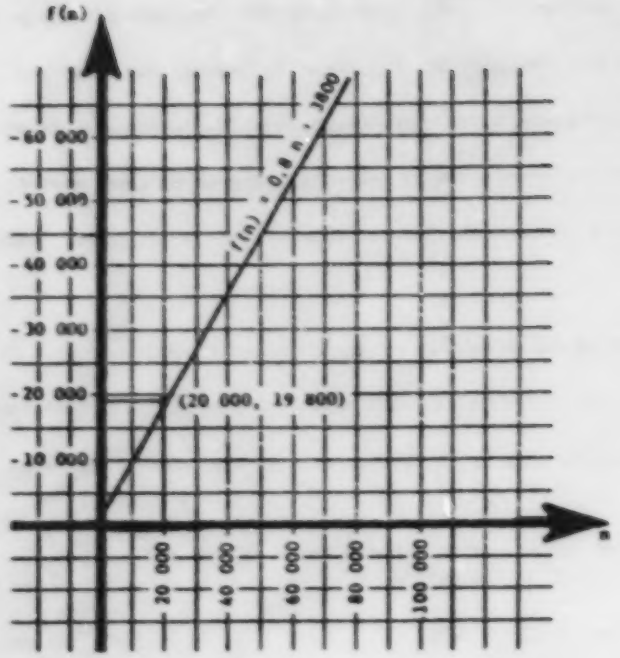
$$f(56\,548{,}667\,72) = 0{,}8 \times 56\,548{,}677\,72 + 3\,800$$
$$= 49\,038{,}934\,21.$$

In order to be able to cross the bridge safely, the maximum load that the truck can carry is:

$$f(n) = 35\,000$$
$$0{,}8n + 3\,800 = 35\,000$$
$$0{,}8n = 31\,200$$
$$n = 39\,000$$

Therefore the truck can safely carry up to 39 000 litres of oil in its tank.

If the tanker contains 20 000 litres of oil, the model makes it possible to determine that the total mass of the truck will be 19 800 kg.



The affine function $f(x) = ax + b$ has many applications. Here are a few examples:

- tension = resistance x intensity
  $E = Ri$ or IR
- force = mass x acceleration
  $f = m\,a$
- sales price = unit price x mass
  $s = u\,m$
- discount = reduction rate x marked price
  $d = r\,a$
- annual interest = capital x interest rate
  $i = c\,r$

Many mathematical problems dealing with selling and buying are often considered as applications of affine functions. Without entering into too many details, it should be pointed out that there are certain differences that link these problems either to a step function or to an affine function throughout a given interval.

Here is an example that illustrates the latter case.

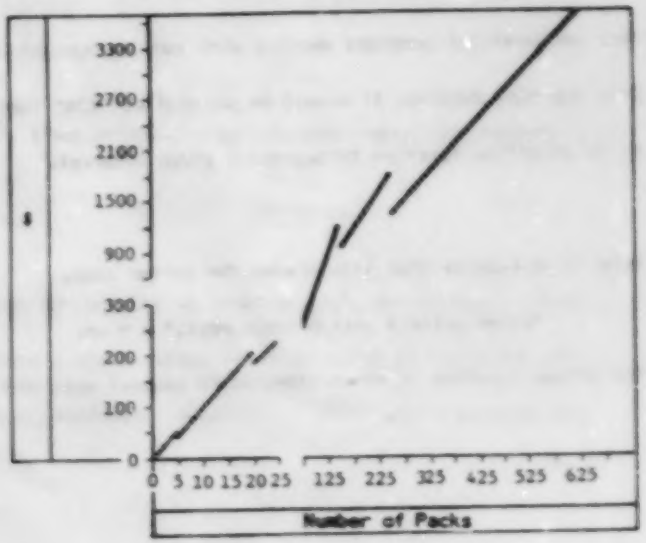
"sales price = unit price x mass," $s = um$,

The affine function is often affected in unusual ways when large quantities are involved.

Here are some data pertaining to this situation

Unit Prices	Quantities
$13.45	0 - 4 packs
$11.50	1 - 5 cartons (1 carton contains 4 packs)
$9.80	5 - 24 cartons
$7.85	24 - 48 cartons
$5.97	1 - 3 tonnes (1 tonne contains 48 cartons)

The graph on the right illustrates this situation



D. Equations

The expression $aX = c$ is equivalent to $aX + by = c$ for $b = 0$. However, the graphs of the two functions make it possible to visualize a clear difference between them.

In one case the solution set is a singleton (a point), and in the other case it is a set containing an infinite number of elements (a line).

The techniques used for solving a system of equations are different from those used for solving an equation with one unknown. These techniques are also very different from one another. On the following pages we have a system whose solution is sought by means of a number of methods, specifically: reduction, substitution, comparison, Cramer's rule, matrices, repetition. This note completes the note already presented in connection with Intermediate Objective 1.4.6.

Consider the system: $4X + y = 23$ (1)
$2X + 3y = 19$ (2)

(a) The reduction method

Multiply (2) by -2:

$$-4X - 6y = -38 \qquad (3)$$

Add (1) and (3):

$$-5y = -15$$

$$y = 3 \qquad (4)$$

Substitute (4) in (1):

$$4X + 3 = 23$$

$$4X = 20$$

$$X = 5$$

The solution is: (5, 3).

(b) The substitution method

(1) becomes:

$$y = 23 - 4X \qquad (3)$$

Substitute (3) in (2)

$$2X + 3(23 - 4X) = 19$$

$$2X + 69 - 12X = 19$$

$$-10X = 150$$

$$X = 5 \qquad (4)$$

Substitute (4) in (3):

$$y = 23 - 4 \times 5$$

$$y = 23 - 20$$

$$y = 3$$

The solution is: (5, 3).

(c) The comparison method

(1) becomes:

$$4x = 23 - y$$

$$x = \frac{23 - y}{4} \qquad (3)$$

(2) becomes:

$$2x = 19 - 3y$$

$$x = \frac{19 - 3y}{2} \qquad (4)$$

(3) and (4) give:

$$\frac{23 - y}{4} = \frac{19 - 3y}{2}$$

$$46 - 2y = 76 - 12y$$

$$10y = 30$$

$$y = 3 \qquad (5)$$

Substitute (5) in (3):

$$x = \frac{23 - 3}{4}$$

$$x = \frac{20}{4}$$

$$x = 5$$

The solution is (5, 3).

(d) Cramer's method

(1) and (2) give:

$$\begin{matrix} 4 & -1 & 23 \\ 2 & 3 & 19 \end{matrix} \qquad 4 \times 3 - 1 \times 2$$

$$\begin{matrix} 4 & 1 & - & 23 \\ 2 & 3 & & 19 \end{matrix} \qquad 23 \times 3 - 1 \times 19$$

$$\begin{matrix} 4 & 1 & - & 23 \\ 2 & 3 & & 19 \end{matrix} \qquad 4 \times 19 - 23 \times 2$$

It is possible to write:

$$x = \frac{23 \times 3 - 1 \times 19}{4 \times 3 - 1 \times 2} = \frac{69 - 19}{12 - 2} = \frac{50}{10} = 5$$

$$y = \frac{4 \times 19 - 23 \times 2}{4 \times 3 - 1 \times 2} = \frac{76 - 46}{12 - 2} = \frac{30}{10} = 3$$

The solution is: (5, 3).

(e) The matrix method

(1) and (2) give:

$$\begin{bmatrix} 4 & 1 & 23 \\ 2 & 3 & 19 \end{bmatrix}$$

Multiply (1) by 3 and subtract from (2):

$$\begin{bmatrix} -10 & 0 & -50 \\ 2 & 3 & 19 \end{bmatrix}$$

Divide (1) by -10

$$\begin{bmatrix} 1 & 0 & 5 \\ 2 & 3 & 19 \end{bmatrix}$$

Divide (2) by 2 and subtract from (1):

$$\begin{bmatrix} 1 & 0 & 5 \\ & -\frac{3}{2} & -\frac{9}{2} \\ 0 & & \end{bmatrix}$$

Multiply (2) by -2/3:

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$

And we have:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \text{ because } \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

The solution is: (5, 3).

To solve any system of equations, it is necessary to operate on columns or rows of data usually presented in the form of tables, graphs, or grids. These modes of presentation of data are known as matrices. The effective method of processing such sets of elements arranged in rows or columns is called matrical algebra.

In order to solve problems by means of matrices, one should be able to represent a graph by a matrix and vice-versa, to represent a relation by a matrix and vice-versa, and to represent a situation by a matrix and vice-versa. One should also be able to understand the meaning of the elementary terms associated with matrices (square matrix, matrix of order n x n, etc). Similarly, one should be able to operate on matrices using elementary transformations:

- interchanging,
- multiplying a row by a non-zero real number,
- adding or subtracting two rows.

It is equally important to know how to find the product of a matrix and a real number or the product of a matrix and a matrix.

(f) The repetitive method

The pocket calculator can be used to teach the pupil another solving method, namely, the repetitive method.

Here is an example:

(1) becomes: $y = 23 - 4x$ (3) (2) becomes: $2x = 19 - 3y$

$$x = \frac{19 - 3y}{2} \quad (4)$$

The following procedures are used:

0 STO

23 − 4 × RCL STO and (19 − 3 × RCL) ÷ 2 STO

In the following table, it is sufficient to pose the hypothesis that $x = 0$ in (3) and to start the repetition process:

(3) in (4), (4) in (3), (3) in (4), (4) in (3), and so on.

Steps	x		y
1	0,0000		23,0000
2	-25,0000		123,0000
3	-175,0000		723,0000
4	-1075,0000		4323,0000
5	-6475,0000		25 923,0000

This table shows clearly that the successive values of x and y do not tend respectively toward a "certain value."

To solve this problem, it is sufficient to isolate the variables "y" and "x", using equations (2) and (1) respectively (as opposed to (1) and (2) in the preceding example). Thus:

(2) becomes:

$$3y = 19 - 2x$$

$$y = \frac{19 - 2x}{3} \qquad (5)$$

(1) becomes:

$$x = \frac{23 - y}{4} \qquad (6)$$

The following procedures are used:

0 [STO]

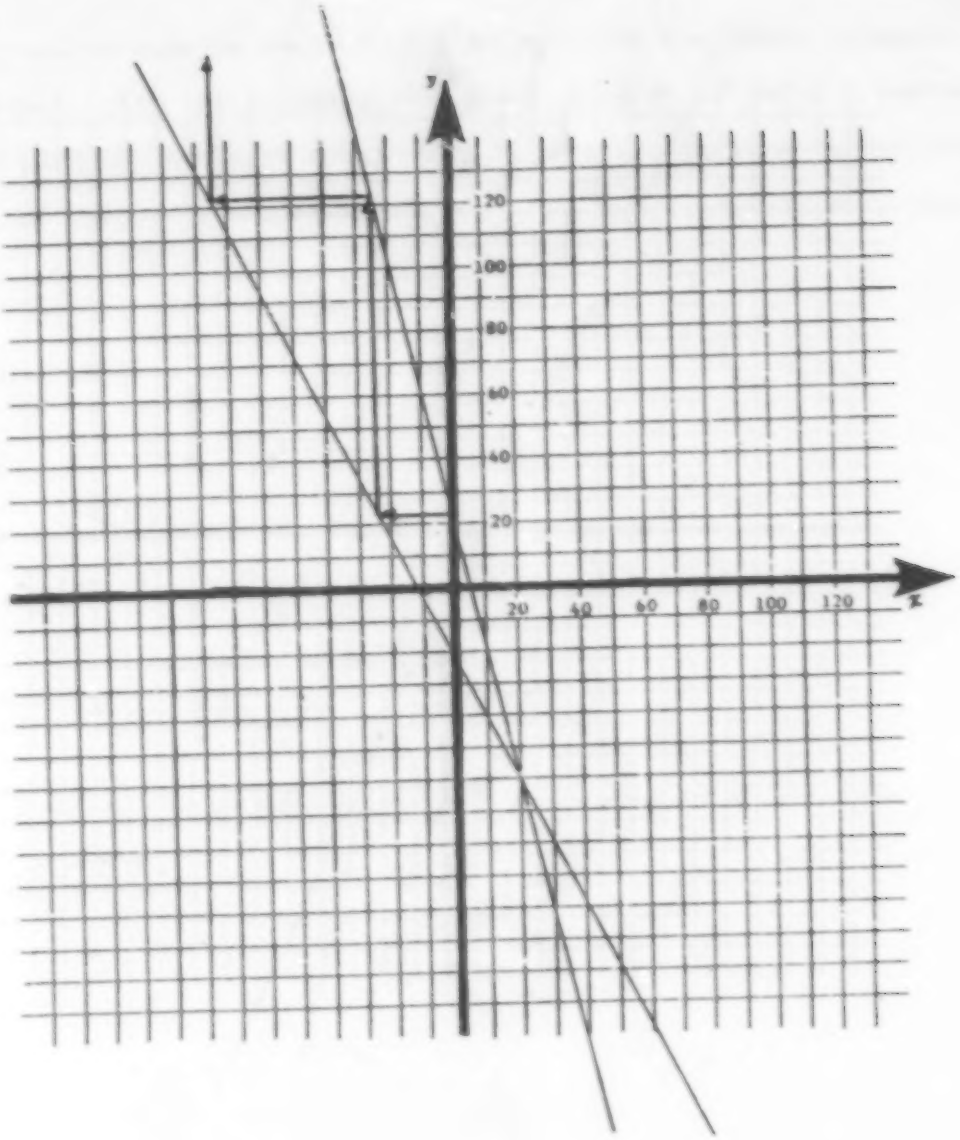
[(] 19 [−] 2 [×] [RCL] [)] [÷] 3 [STO] and [(] 23 [−] [RCL] [)] [÷] 4 [STO]

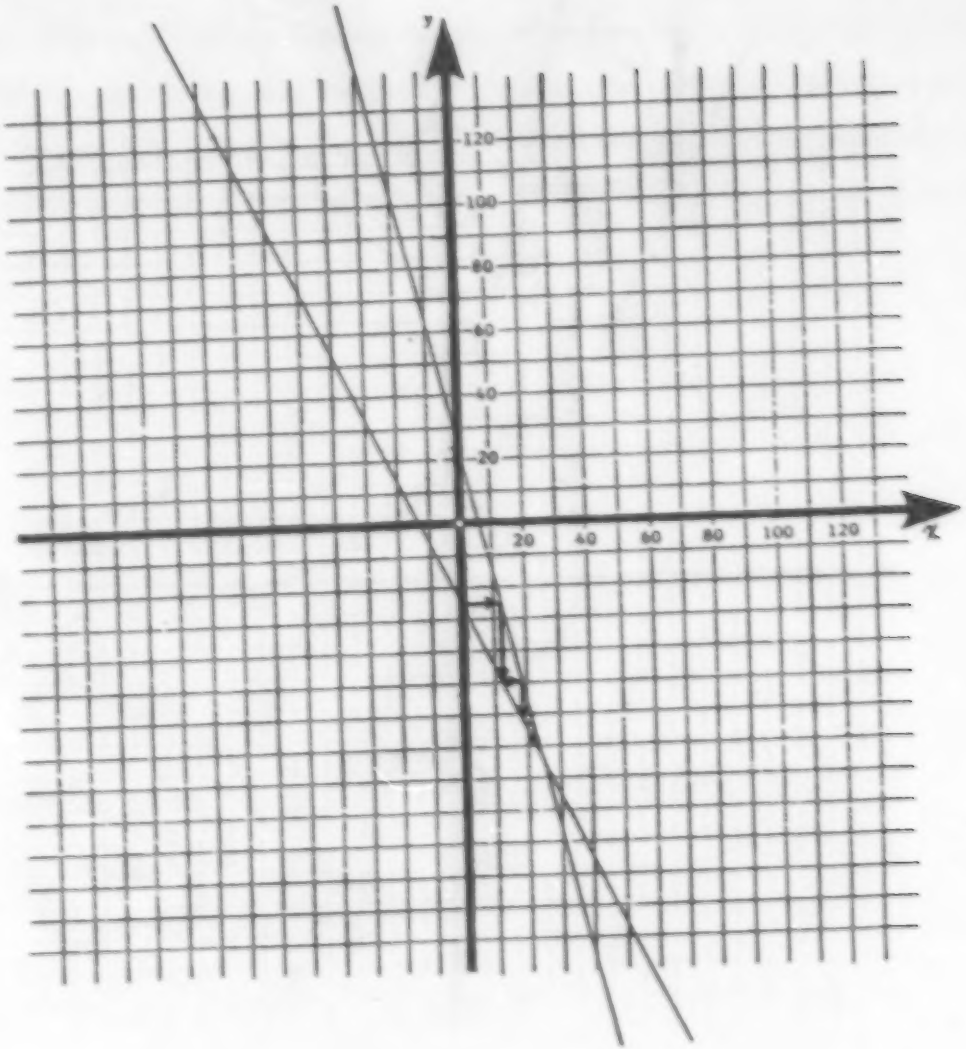
In the next table, it is sufficient to pose the hypothesis that $X = 0$ in (5) and to start the repetitive process. The table shows clearly that X and y tend respectively toward 5 and 3.

Steps	X		y
1	0,0000		6,3330
2	4,1667		5,5556
3	4,8611		3,0926
4	4,9769		3,0154
5	4,9961		3,0026

Starting from 0 or from an approximate value, and executing the sequence of procedures several times in a row, is a repetitive process. In the preceding example, it is necessary to check the validity of the process, regardless of the initial value. For instance, the pupil should try the following values: 100, -150, 0,001, 1585, etc.

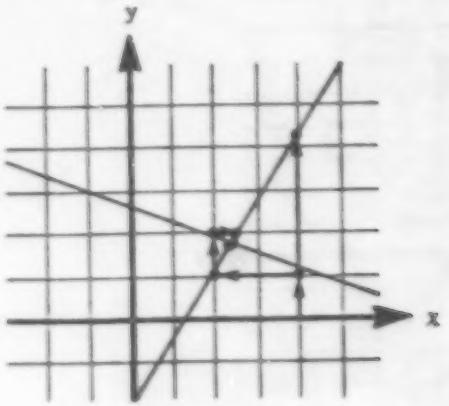
To help the pupil understand how the repetitive process works, the teacher should ask him to enter on the graph each of the values found in the two preceding tables. The arrows illustrate the sequence in which the values were found.


y
120
100
80
60
40
20
20
40
60
80
100
120
x


y
120
100
80
60
40
20
20
40
60
80
100
120
x

We have two sets of "steps," one that moves away from the solution, and another that moves towards it, regardless of the initial value selected (as long as this value is not 5). When the product of the rates of change of each function is negative, we no longer have a set of steps but a coiling spiral (one that moves towards the solution) or an uncoiling spiral (one that moves away from the solution). The same principle that is at work in the set of steps applies in this case. For example, the following graph applies to the system:

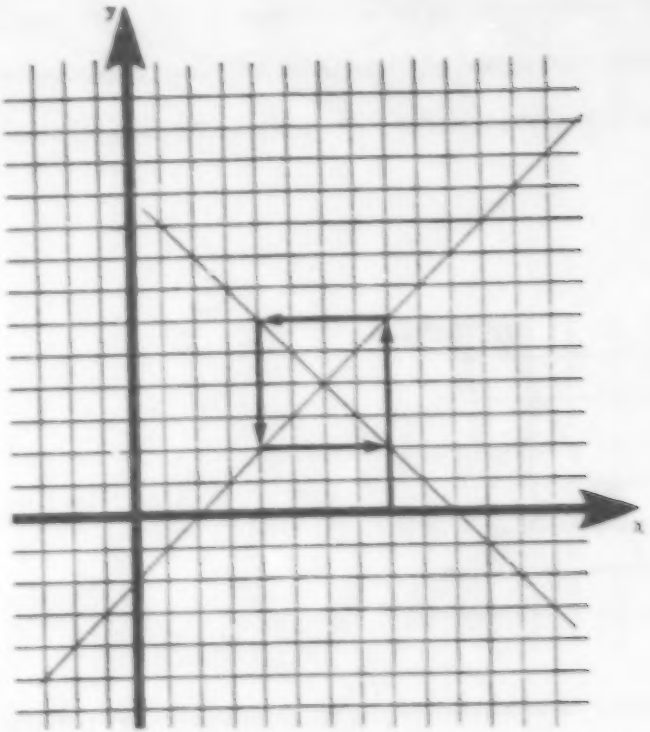
$$x - y = 2$$
$$3x + 2y = 3$$



The teacher might consider it worthwhile for the pupil to study certain special cases such as the following:

$$x + y = 5 \qquad x - 2y = -8$$
$$2x - 2y = 3 \quad \text{or} \quad 3x - 6y = -12$$

In the first case, we turn in a circle since we always return to the starting point.

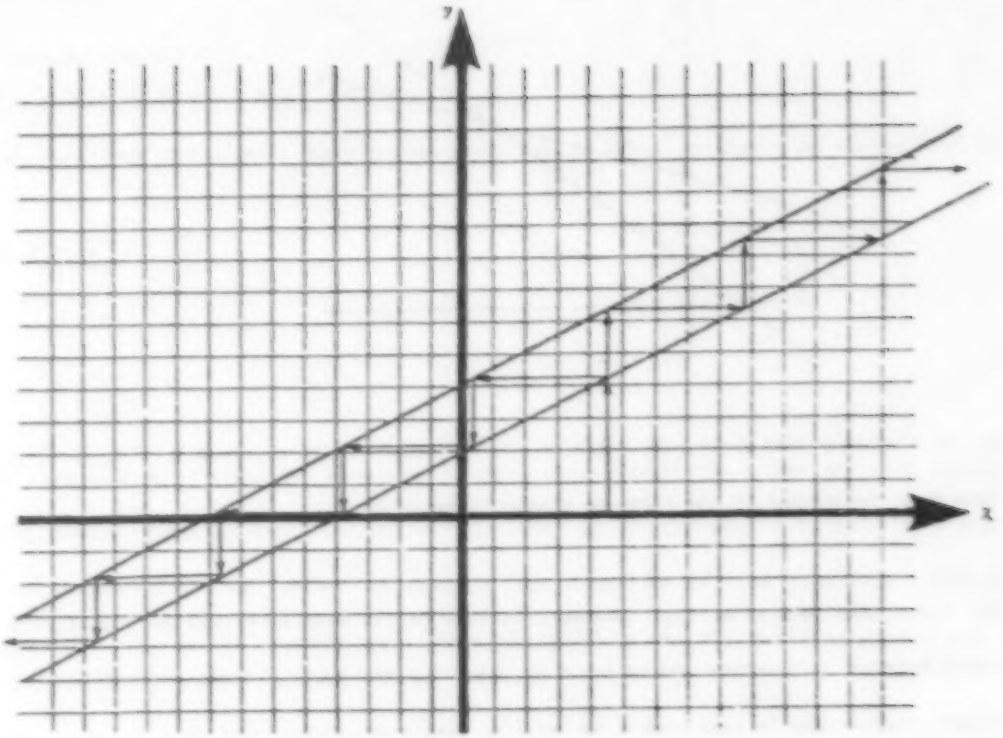


The product of the rates of change is characteristic. To find the procedures that make it possible to arrive at the solution, we have only to use an equivalent system such as:

$x + y = 5$

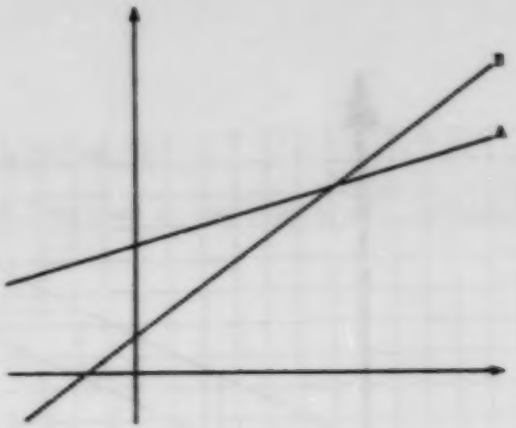
$3x - y = 8$ (sum of the equations)

In the second case, the set of steps is regular; the steps all have the same height.



The rates of change are identical.

It is advisable to vary the rates of change and to have the pupil experiment with different systems. Using the graph of a given system, have the pupil find the rules underlying the procedures to be applied in order to arrive at the solution. Here is an example:



If A and B are the procedures for finding y from X for each function, we will execute the sequence A, B^{-1}, A, B^{-1},..., to move towards the solution. Such is the case when the product of the rates of change is positive.

The repetition method may not always seem very effective. However, in view of its simplicity and the possibility of applying it when interpreting graphs, a basic knowledge of this method should prove an asset for the pupil. These are sufficient reasons for teaching the repetition method. Moreover, this method is also useful for solving simple equations, such as $X = \cos X$, for which there are no formulas. As this method requires long calculations, it is obviously necessary to use an electronic calculator.

The solution of an equation can be found by means of operations using the repetition method. For example, 108 + x = (180 + x) x 0,76 can be solved by composing this system:

$$y = 108 + x$$
$$y = (108 + x) \times 0{,}76,$$

and by performing these procedures:

[x] 0,76 [=] [+] 136,8 [=] followed by [-] 108 [=]

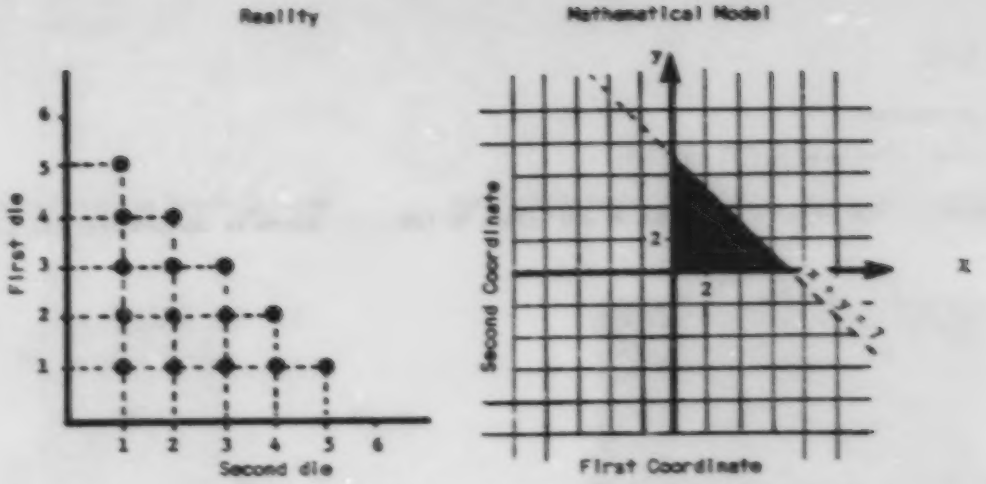
Using a random value, we notice that after a few steps we come close to 120.. It should be obvious that the sequence of procedures can be simplified:

[x] 0,76 [+] 28,8 [=]

E. Inequations

Graphing is a useful technique for finding the solutions of one or more inequations. Consider such familiar situations as buying lottery tickets, drawing tokens, throwing dice, and so on. All these can be represented by means of a graph often located only in the first quadrant or on the points of the "grid." A selection of situations of this type should help the pupil realize the broad applications of algebra and could be the starting point for a discussion on the limits within which graphs can be interpreted.

Example: Determine the situations in which the sum of two dice is lower than 7.

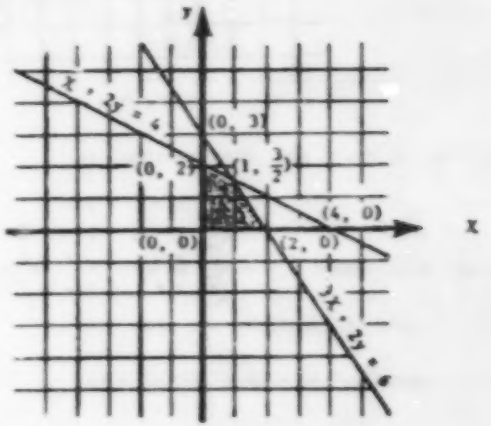


Linear programming is another subject that contributes to the development of mathematization and problem-solving skills. In business and industry, linear programming facilitates decision-making, the definition of maximum returns, of minimum costs, and so on.

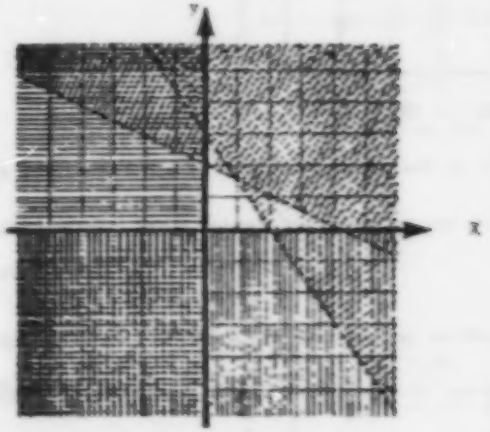
In linear programming there are two ways of representing a convex polygon. Consider the following model:

$$x \geq 0$$
$$y \geq 0$$
$$3x + 2y \leq 6$$
$$x + 2y \leq 4$$

(a) Crosshatches representing the convex polygon:



(b) Crosshatches representing the complementary regions:

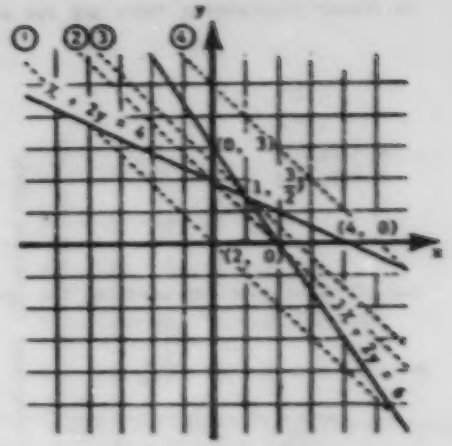


When evaluating an expression of the type $z = ax + by + c$ for the values of the coordinates of the points of a convex polygonal region, it becomes obvious that both the highest and the lowest values are found on at least one of the vertices.

The line which corresponds to $z = ax + by + c$ is an element of the family of direction lines $-\frac{a}{b}$.

In the graph on the right, $z = x + y$ has a maximum value of 2,5 at the point $(1, \frac{3}{2})$ of the convex polygon. $z = x + y = 5$ is a value that does not belong to the convex polygonal region under consideration.

The optimization points can be found by applying the following theorem: "The evaluation of the expression $z = ax + by + c$ by the values of the coordinates of the points of a convex polygonal region is maximum at least at one of the vertices." It can be concluded that if two vertices maximize the expression, then all the points on the segment linking these two vertices are maxima.



① $z = x + y = 0$
② $z = x + y = 2$
③ $z = x + y = 2,5$
④ $z = x + y = 5$

Consider the following situation: "There are 6 red tokens and 4 yellow ones in a bag."

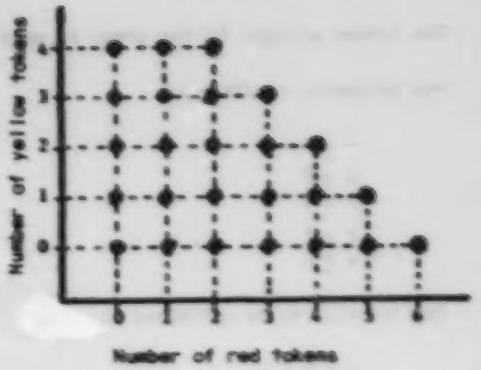
Question 1: If we draw 6 tokens at the most from the bag, how many reds and how many yellows are we likely to draw?

Let r be the number of red tokens and y the number of yellow ones.

The model:

$$r \geq 0$$
$$y \geq 0$$
$$r \leq 6$$
$$y \leq 4$$
$$r + y \leq 6$$

is represented by the graph on the right.



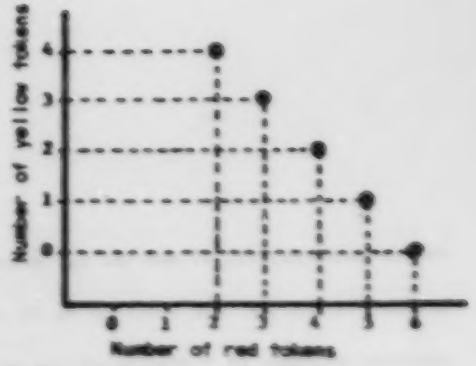
Question 2: If we draw exactly 6 tokens from the bag, how many yellows and how many reds are we likely to draw?

Let r be the number of red tokens and y the number of yellow ones.

The model:

$$r \geq 0$$
$$y \geq 0$$
$$r \leq 6$$
$$y \leq 4$$
$$r + y = 6$$

is represented by the graph on the right.



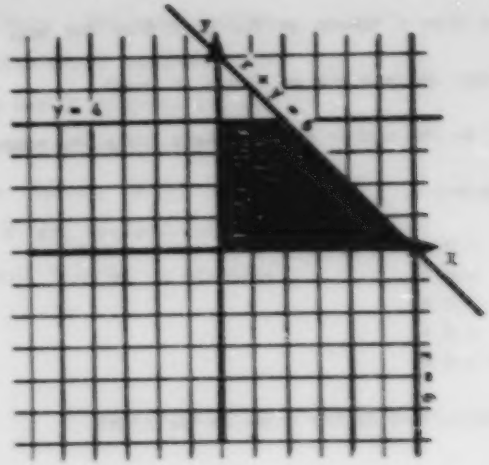
It should be noted that the mathematical model is much more general and covers more than the reality described in the statement of the problem. The graph reproduced below represents the general model associated with the situation under consideration.

The convex polygon in the graph is associated with the following conditions:

$$r \geq 0$$
$$y \geq 0$$
$$r \geq 6$$
$$y \geq 4$$
$$r + y \geq 6$$

The function to be optimized is

$$z = r + y$$



The optimal situation is found by substituting the coordinates of the vertex in the function to be optimized:

Vertices of the Polygon	Values of z
(0, 0)	0
(0, 4)	4
(2, 4)	6
(6, 0)	6

As two vertices, (2, 4) and (6, 0), allow a maximum value of 3, we conclude that the segment linking these two vertices also give the optimal values for the function to be optimized. The final conclusion is that (2, 4), (3, 3), (4, 2), (5, 1), and (6, 0), are all possible outcomes of a drawing of red and yellow tokens.

Here is another situation that illustrates the usefulness of graphs for finding the solution to a given problem.

There are 200 birds on a farm, a mixed flock of turkeys, roosters, and hens. The owner wishes to sell at least 50 turkeys and 10 roosters, but no more than 100 turkeys and 20 roosters and not more than 100 birds of both kinds. If his net profit is $3.50 for a turkey, $3.00 for a rooster, and $2.00 for a hen, how many of each kind of bird should he sell in order to maximize his profits?

Let x be the number of turkeys,

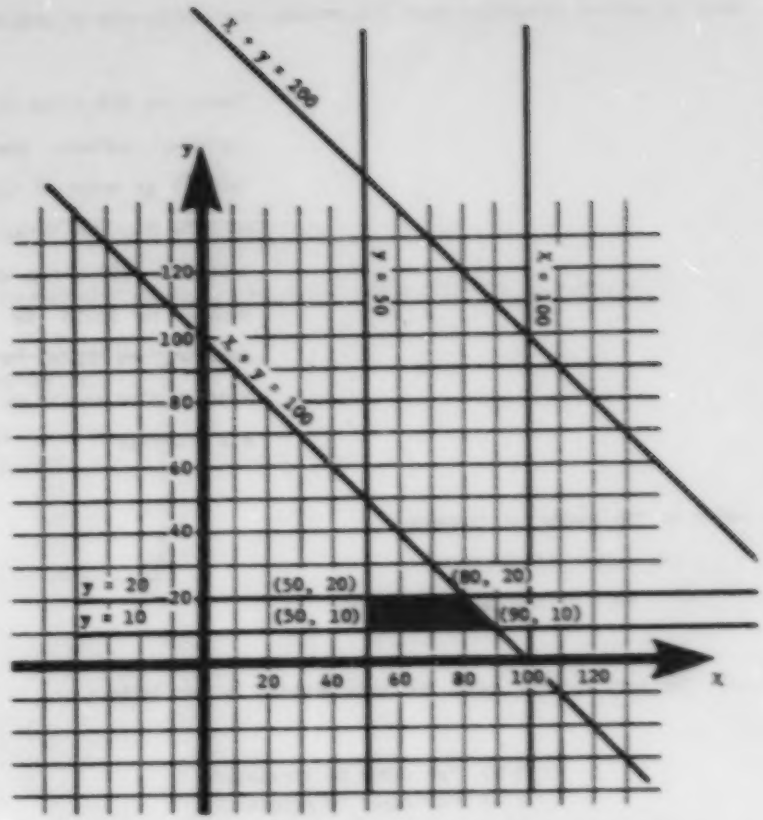
y the number of roosters,

$200 - (x + y)$ the number of hens.

(a) The mathematical model for this problem is the following:

$$
\begin{aligned}
x &\geq 50 && \text{"at least 50 turkeys;"}\\
y &\geq 10 && \text{"at least 10 roosters;"}\\
200 - (x + y) &\geq 100 && \text{it is not possible to have a negative number of hens;}\\
x &\leq 100 && \text{"no more than 100 turkeys;"}\\
y &\leq 20 && \text{"no more than 20 roosters;"}\\
x + y &\leq 100 && \text{"no more than 100 of both."}
\end{aligned}
$$

(b) The graph of this model is the following:



The shaded section represents the convex polygon determined by the intersection of the six half-planes associated with the inequations of the model. It is also possible to determine the exact coordinates of the vertices of the convex polygon:

$x = 50$ and $y = 10$ give (50,10)
$x = 50$ and $y = 20$ give (50,20)
$x + y = 100$ and $y = 10$ give (90,10)
$x + y = 100$ and $y = 20$ give (80,20)

The function to be optimized is:

$$z = 3{,}5x + 3y + 2[200 - (x + y)]$$

$$z = 3{,}5x + 3y + 400 - 2x - 2y$$

$$z = 1{,}5x + y + 400$$

The maximum returns on the sale can be found by substituting the coordinates of the vertices in the function to be optimized:

Vertices of the Polygon	Values of z
(50, 10)	485
(10, 20)	495
(90, 10)	545
(80, 20)	540

(f) Interpreting the result:

The function z is maximum at the vertex (90,10). Therefore, there are 90 turkeys, 10 roosters, and

200 - (90+10) = 200 - 100 = 100 hens.

Here is another example:

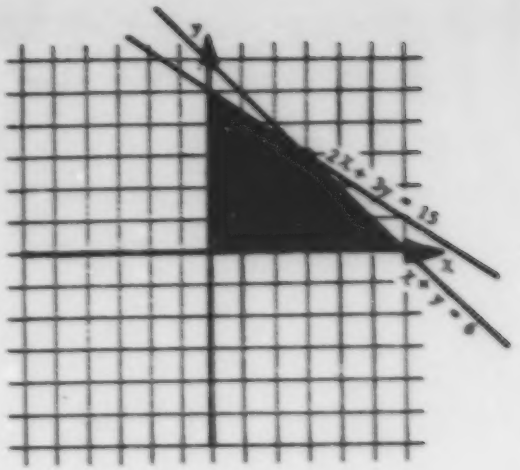
A company uses computers A and B in shared time for solving certain environmental problems, namely, air and water pollution. The company has access to computer A for only six minutes per hour, and to computer B for only fifteen minutes per hour. A water pollution problem requires the use of computer A for one minute, and that of computer B for two minutes. An air pollution problem requires the use of computer A for one minute, and that of computer B for three minutes. The company makes a net profit of $250.00 for a research project on water pollution, and a net profit of $400.00 for a research project on air pollution. Find the most profitable time-sharing scheme.

Let x be the number of water pollution problems and y the number of air pollution problems.

If we analyze the situation, we come up with the following constraints:

$x \geq 0$	A negative number of problems is impossible.
$y \geq 0$	A negative number of problems is impossible.
$1x - 1y \leq 6$	"A water pollution problem requires the use of computer A for one minute, and an air pollution problem also requires the use of computer A for one minute."
$2x - 3y \leq 15$	"A water pollution problem requires the use of computer B for two minutes, and an air pollution problem requires the use of computer B for three minutes."

The graph of this model is the following:



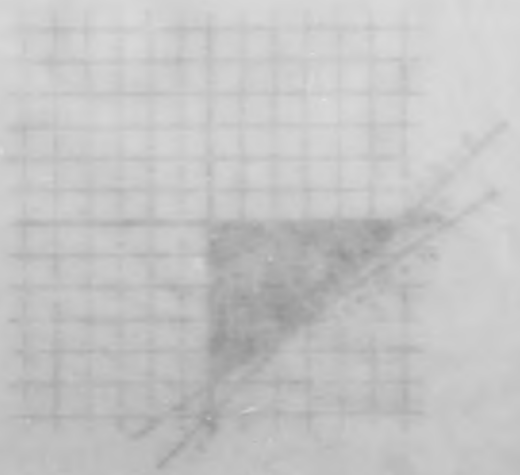
The function to be optimized is:

$$z = 250x + 400y$$

The most profitable scheme is found by substituting the coordinates of the vertices of the convex polygon in the function to be optimized.

Vertices of the Polygon	Values of z
(0, 0)	0
(1, 5)	2000
(3, 3)	1950
(6, 0)	2500

The conclusion, therefore, is that the most profitable solution for the company is to handle only air pollution problems.



EDUCATIONAL EVALUATION

This section of the curriculum guide describes the criteria and techniques which are applicable to the evaluation of learning and which are likely to facilitate the assessment of the extent to which course objectives have been achieved.

It is important to recall the responsibility of the teacher in this regard: "Evaluation takes place first of all in the classroom and ... the prime responsibility is the teacher's."[1] One of the teacher's main concerns should be to integrate educational evaluation into a dynamic learning experience for the pupil.

> * The evaluation of a pupil's learning should be concerned with those instructional objectives which are the subject of his daily efforts. If the evaluation is to bear upon the instructional objectives, it should also be appropriate to the mental processes required by the instruction. This is a question of both quality of education and of justice for the pupil. It would demonstrate incoherence if, for example, the instruction involved comprehension and analysis, while the evaluation, in its turn, was concerned only with simple memorization.[2]

Educational evaluation must always have a reason:

> * Whatever may be the reasons for evaluating a pupil's learning, the teacher must always be aware of the underlying reasons for every evaluation operation he undertakes. Is it to establish the pupil's individual progress toward objectives which have been set? Is it to make a general assessment of the progress of the whole school or of a group taking the course? Is it to form working sub-groups so that instruction can be better adapted to the current learning rates of each group? Is it to provide parents with information about the progress of their children? Is it to assess the effectiveness of the teaching strategies that are being used?[3]

---

1. Québec, Ministère de l'Éducation. The Schools of Québec: Policy Statement and Plan of Action, 1979, p. 94.

2. Québec, Ministère de l'Éducation. General Policy for Educational Evaluation, 1981, No. 16-7500 A, p. 10.

3. Ibid.

Since the teacher is primarily responsible for the educational evaluation of the pupil's learning progress, it is his responsibility to decide at what time a given measurement instrument should be used.

> "Where formative evaluation is involved, the teacher is the best situated to decide when information should be obtained. The teacher knows when the instructional activity has been completed and at what point it would be most appropriate to check if the desired pupil learning has actually taken place.
>
> The teacher or the school principal may also decide at which times summative evaluation would be appropriate. If, for example, it is desirable to determine the level of mastery of a group of pupils for a set of concepts or skills, a group evaluation profile could be made at what is considered to be the appropriate point in the evaluation plan."[1]

In a dynamic learning process, evaluation plays a crucial role. It:

- "provides information on the quality of the pupils' academic performance and general development;
- helps in the appraisal of the quality of the pupils' learning progress;
- helps in the diagnosis of the learning or general development problems of individual pupils or of groups of pupils;
- leads to decisions which make it possible to modify learning activities."[2]

---

1. General Policy for Educational Evaluation, p. 11.
2. S.G.M.E. La mesure critériée et la mesure normative, 1976.
   (This citation is a free translation).

Assignments, control questionnaires, tests, and exams are some of the measurement instruments frequently used in the classroom. Some teachers go further and develop observation checklists as a type of evaluation tool. Along the same lines, this curriculum guide provides forms which are designed to facilitate the carrying out of formative evaluation supported by criterion-referenced interpretation (Form A), while paying due attention as well to summative evaluation, in which norm-referenced interpretation is used (Form B).

Whether or not the forms provided in this document are used, as a general rule the teacher should not take the scores which have been used in a formative evaluation as the basis for a norm-referenced interpretation. These scores are indices which are intended to enable him to take remedial measures and should not normally appear on the pupil's official school report.

The purpose of the summative evaluation recorded on the pupil's school report is to assess the extent to which he has mastered a set of learning objectives. It complements formative evaluation, adding data concerning time (retention) and the accumulation of knowledge (which can in some cases increase the level of mastery).

This type of evaluation involves a minimum standard (the passing mark). Because of these features, it lends itself to criterion-referenced interpretation and can sometimes be used for this purpose.

The following four-part table situates the dynamic process of evaluating a pupil's learning and of interpreting that evaluation within the context of the instructional approach detailed in the preceding paragraphs.

		INTERPRETATION	
		NORM-REFERENCED	CRITERION-REFERENCED
EVALUATION	Formative	FOR OCCASIONAL USE	FOR FREQUENT USE
	Summative	FOR FREQUENT USE	FOR OCCASIONAL USE

Form A makes it possible to effect a formative evaluation of learning. Results can be measured through diagnostic tests, which ensure measurement of the extent to which each terminal objective has been mastered. These tests lend themselves to a criterion-referenced interpretation of the learning progress of each pupil and help the teacher take the necessary remedial measures. Such tests are for diagnostic purposes only and should not be used for decisions regarding promotion. The degree of mastery of each general objective is evaluated separately and is expressed as a score over 100, which is then weighted by the factor n. Weighting is necessary so that the mark reflects the relative importance accorded to each general objective. The value of n can be determined, using as a basis the percentages shown in the table(s) in Section 5.2 of the program. These values range from 0 to 100%. The mark obtained in this way is entered in the appropriate box in Form B.

Since a pupil's progress is not always directly proportionate to his scores, the teacher is strongly encouraged to carry out a co-evaluation in which both he and the pupil assess the latter's general progress with respect to generally accepted criteria: effort, perseverance, quality and quantity of work, participation, and so on. The result is expressed using a scale such as the one which follows:

5 excellent,
4 very good,
3 good,
2 average,
1 below average,
0 unsatisfactory.

The average of these co-evaluations is entered in the corresponding box on Form B. After Form A has been completed, the teacher can use it as a tool to carry out a criterion-referenced evaluation of each pupil's learning.

Form B makes it possible to compile and compare the marks which have been obtained through co-evaluations and through exams, and which have been calculated on each of the A-type forms. The average of the marks in the co-evaluation column and the sum of the weighted marks in the examination column makes it possible to form an overall picture of the pupil's progress.

The mark obtained on the final examination, the content of which must cover all the general objectives of the program, makes it possible to ascertain whether or not the pupil has been able to synthesize the concepts studied. The final examination must be prepared and administered by the school body (the school board, the regional school board, or the ministère de l'Éducation) which is responsible for implementing this program. A mark out of 50 for tests, combined with a mark out of 50 for the final exam, provides a score expressed out of 100 which makes it possible to make a general assessment of the pupil's learning progress. The two marks appearing at the bottom of the form are important elements in summative evaluation with regard to the academic knowledge acquired by the pupil and in decisions concerning his promotion.

The following pages illustrate Forms A and B, which can be used for the entire first cycle program of secondary school.

FORM A

First Name:	Surname:
Program: Second Cycle, Sec. III	

GENERAL OBJECTIVE	N° 1	TO ENCOURAGE pupils to apply their knowledge of arithmetic and algebra.

	TERMINAL OBJECTIVES	Co-evaluation	Diagnostic Test	Examination
1.1	TO SOLVE real-life problems involving real numbers.			
1.2	TO SOLVE real-life problems involving an algebraic equation (or inequation) of the first degree in one real variable.			
1.3	TO SOLVE problems of everyday life involving a constant or affine function.			
1.4	TO SOLVE problems arising from situations involving a system of equations of the first degree in two variables.			
1.5	TO SOLVE problems arising from situations involving a system of inequations of the first degree in two real variables.			

A

n = 0,40

$\frac{}{100} \times n$

Average

Co-evaluation: 5 excellent
4 very good
3 good
2 average
1 below average
0 unsatisfactory

Remarks: _____

FORM B

First Name:	Surname:
Program: Second Cycle, Sec. III	

	GENERAL OBJECTIVES	Co-evaluation	Examination	Final Examination
1	TO ENCOURAGE pupils to apply their knowledge of arithmetic and algebra.			
2	TO ENCOURAGE pupils to analyze geometric situations.			
3	TO INTRODUCE pupils to the analysis of statistical and probability data.			

B

Co-evaluation: 5 excellent
4 very good
3 good
2 average
1 below average
0 unsatisfactory

Promotion: _____

Remarks: _____

_____

_____

Average

100
100
200
100

First Name:	Surname:
Program: Second Cycle, Sec. IV	

GENERAL OBJECTIVE	N° 1	TO ENCOURAGE pupils to apply their knowledge of arithmetic and algebra.

	TERMINAL OBJECTIVES	Co-Evaluation	Diagnostic Test	Examination
1.1	TO SOLVE real-life problems involving real numbers.			
1.2	TO SOLVE real-life problems involving an algebraic equation (or inequation) of the first degree in one real variable.			
1.3	TO SOLVE problems of everyday life involving a constant or affine function.			
1.4	TO SOLVE problems arising from situations involving a system of equations of the first degree in two variables.			
1.5	TO SOLVE problems arising from situations involving a system of inequations of the first degree in two real variables.			

A

n = 0,40

x n

$\frac{}{100}$

Average

Co-evaluation: 5 excellent
4 very good
3 good
2 average
1 below average
0 unsatisfactory

Remarks: _____

FORM B

First Name:	Surname:
Program: Second Cycle, Sec. IV	

	GENERAL OBJECTIVES	Co-evaluation	Examination	Final Examination
1	TO ENCOURAGE pupils to apply their knowledge of arithmetic and algebra.			
2	TO ENCOURAGE pupils to analyze geometric situations.			
3	TO INTRODUCE pupils to the analysis of statistical and probability data.			

B

Co-evaluation: 5 excellent
4 very good
3 good
2 average
1 below average
0 unsatisfactory

Promotion: _____

Remarks: _____

_____

_____

Average

100

100

200

100

FORM A

First Name:	Surname:
Program: Second Cycle, Sec. V	

GENERAL OBJECTIVE	Nº 1	TO ENCOURAGE pupils to apply their knowledge of arithmetic and algebra.

	TERMINAL OBJECTIVES	Co-Evaluation	Diagnostic Test	Examination
1.1	TO SOLVE real-life problems involving real numbers.			
1.2	TO SOLVE real-life problems involving an algebraic equation (or inequation) of the first degree in one real variable.			
1.3	TO SOLVE problems of everyday life involving a constant or affine function.			
1.4	TO SOLVE problems arising from situations involving a system of equations of the first degree in two variables.			
1.5	TO SOLVE problems arising from situations involving a system of inequations of the first degree in two real variables.			

A

n = 0,46

$\frac{\quad}{100} \times n$

Average

Co-evaluation: 5 excellent
4 very good
3 good
2 average
1 below average
0 unsatisfactory

Remarks: _____

FORM B

First Name: Surname:

Program: Second Cycle, Sec. V

	GENERAL OBJECTIVES	Co-evaluation	Examination	Final Examination
1	TO ENCOURAGE pupils to apply their knowledge of arithmetic and algebra.			
2	TO ENCOURAGE pupils to analyze geometric situations.			
3	TO INTRODUCE pupils to the analysis of statistical and probability data.			

B

100
100
Average 200
100

Co-evaluation: 5 excellent
4 very good
3 good
2 average
1 below average
0 unsatisfactory

Promotion: _____

Remarks: _____

_____

_____

INSTRUCTIONAL MATERIALS

INSTRUCTIONAL MATERIALS

The teacher can make the subject of mathematics meaningful by helping the pupil discover for himself, and apply, the following basic concepts: classification, numbers, numeration, measure, and geometric operations or forms. In such an approach, the actual use of these concepts is of primary importance. It is through everyday situations, motivating learning activities, and a variety of experiments in the form of games, codes, and representations, that the pupil comes to formulate concepts in his own terms and to develop his own logical reasoning skills.

In order to enhance this process, both teacher and pupils should have access to a wide variety of readily available instructional materials. Due to the many constraints imposed by the cost of materials, the number of users, and storage problems, this material should be organized in a pedagogically sound and physically logical manner.

This section provides a list of instructional materials. It is not exhaustive and does not include either written documents such as textbooks, exercise books, answer books, teacher's guides, or audio-visual documents.

CLASSROOM MATERIALS

QUANTITY	DESCRIPTION
	Measurement of Angle and Length
1	Wooden blackboard compass, 420 mm long, double pivot, rubber/steel.
1	Flexible curve, lead bar encased in vinyl mounting, 600 mm long.
1	Plastic or wooden blackboard set square, 30°-60°, two-sided, 500 mm high, graduated, with handle.
1	Plastic or wooden blackboard set square, 45°, two-sided, 400 mm high, graduated, with handle.
1	Plastic or wooden blackboard protractor, 180°, two-sided, 500 mm baseline.
1	Plastic or wooden blackboard ruler, 64 mm wide, 1000 mm long, graduated, with handle.
1	Plastic or wooden T-square, 900 mm long, with handle.
3	Clinometer, revolving type, plastic and lead.
3	Plastic or wooden calibrating compass, 300 mm long for inside and outside measurements.
3	(0-25)mm or 0,01 mm stainless steel micrometer.
3	Sliding callpers, 900 to 1000 mm long, graduated, for inside and outside measurements.
3	Plastic trundle wheels, 1000 mm circumference, graduated, clicking mechanism, handle.
3	Fibreglass tape measures, 20 000 mm long, graduated in mm.

QUANTITY	DESCRIPTION
	Volume and Surface
1	Plastic centimetre cube (1000), 1 g mass, faces of assorted colours, interlocking.
1	Clear plastic litre container, cube with 100 mm edge, graduated in millimetres.
35	Transparent plastic grids, 200 mm x 200 mm, surface measurement.
1	Dismountable meter cube frame, 3 rods painted in decimetres.
1	Plastic beaker, 1000 ml, graduated in 50 ml, with handle and pouring lip.
1	Plastic hollow blocks, 150 mm high, with filling holes, set of 14.
	Geometry and Algebra
1	Centre angle and inscribed angle, magnetic material.
17	Geoboards, 300 mm x 300 mm, pegs, elastic bands of assorted colours.
1	Parallel rule cut by a transversal, magnetic material.
1	Pythagorean theorem, magnetic material.

QUANTITY	DESCRIPTION
	Supplementary Materials
1	Compass, Brunton "Cadet"type, graduations 0-360.
35	Calculators.
1	Numerical straightedge, magnetic blackboard, 200 mm wide, 1500 mm long, disks.
1	Initiation to probability and statistics kit.
1	Inclinable level, horizontal angles, magnification 18x.
1	Compensating polar planimeter, plastic, magnifying lens.
10	Polyforms, assembly of rigid cardboard boxes, elastic bands.
10	Structure building, plastic straws, junction knots.
10	Auto Correct Art, case and sets.
10	Veritech, self-correcting, box.
35	Mirrors (Mira or other).
1	Multi-base blocks (1 set).
2	Logic grids and pattern blocks.
2	Trimath and quadrimath.
1	Polar coordinate and rectangular coordinate board (two-sided).
	A variety of mathematical games.
8	Micro-computers.

MATERIAL SUPPLIED BY THE SCHOOL

QUANTITY	DESCRIPTION
	Wood glue, white, 5 kg container.
	Set of cardboard boxes, assorted colours.
	Graph paper squared in millimetres.
35 per classroom-workshop	All-purpose scissors, steel, 200 mm long, right-handed, left-handed.

MATERIAL BOUGHT BY THE PUPIL

QUANTITY	DESCRIPTION
	Pencil compass, brass, 150 mm long, blocking screw or clamp.
	Professional set square, transparent plastic, 150 mm high, 30°-60°, two-sided.
	Professional set square, transparent plastic, 150 mm, 45°, two-sided.
	Plastic angle protractor, 150 mm, semi-circular, double scale, sighting mark.
	Plastic or wooden ruler, 300 mm, graduated in mm.
	Calculator.

BIBLIOGRAPHY

BIBLIOGRAPHY

This selected bibliography is intended as a reference source for teachers who are looking for ideas for classroom activities.

It should be noted that, as a rule, books approved by the ministère de l'Éducation and materials available from the school boards have not been included in this list of materials.

GENERAL OBJECTIVE 1 TO ENCOURAGE pupils to apply their knowledge of arithmetic and algebra.

A.M.Q. Fascicules: Les Ensembles 1, Les entiers naturels 2, Les systèmes de numération des entiers naturels 3, Les algorithmes pour les opérations sur les entiers naturels 4, Les systèmes de numération des nombres rationnels 5, Les nombres et leurs facteurs 6, Translated from the "SETS" collection, Reston, Virginia: NCTM, 1964.

APSI. Le langage BASIC, 3rd edition. Duvernay (Laval): APSI systèmes informatiques inc., 1978.

AUBÉ, Michel, Benoît COTÉ et al. Programmation en LOGO, guide de travail. PERMAMA, (P984-3016), Télé-université, 1976.

AUBÉ, Michel, Benoît COTÉ et al. Projets en programmation LOGO. PERMAMA, (P984-3017), Télé-université, 1976.

BANWELL, Colin S. et al. Starting Points for Teaching Mathematics in Middle and Secondary Schools. Oxford: Oxford University Press, 1972.

BELLEHUMEUR, M., J.P. VALARD. Eléments d'informatique et programmation fortran. Montreal: Lidec, 1974.

BERGAMINI, David. Mathematics. New York: Time-Life, 1963.

BERGERON, A., J. BORDIER, and M. WARISSE. Utilisation de la calculatrice pour l'enseignement des mathématiques au secondaire, recueil de problèmes. PERMAMA, (P984-5026), Télé-université, 1981.

BERGERON, A., J. BORDIER, and M. WARISSE. Utilisation de la calculatrice pour l'enseignement des mathématiques au secondaire, reference texts. PERMAMA, (P984-5026), Télé-université, 1981.

BERLOQUIN, Pierre. 100 jeux alphabétiques. Paris: Le livre de poche, 3519, 1973.

BERLOQUIN, P. 100 jeux et casse-tête. Paris: Le livre de poche, 4165, 1975.

BERLOQUIN, P. 100 jeux logiques. Paris: Le livre de poche, 3568, 1973.

BERLOQUIN, P. 100 jeux numériques. Paris: Le livre de poche, 3669, 1973.

BERLOQUIN, P. Testez votre intelligence. Paris: Le livre de poche, 3915, 1974.

BORDIER, J. et al. Entiers relatifs et nombre rationnels. PERMAMA, (P984-3030), Télé-université, 1977.

BOUCHARD et al. Modèles mathématiques et réalité. PERMAMA, (P184-4004), Télé-université, 1978.

BOUCHARD et al. Modèles mathématiques et réalité, niveaux d'aide. PERMAMA, (P184-4004), Télé-université, 1978.

BOUCHER, Claude. Les nombres, leur histoire. Collection Les mathématiques aujourd'hui, Montréal: Lidec, 1965.

BOUVIER, A. et al. Dictionnaire des mathématiques. Paris: Presses universitaires de France, 1979.

BOYER, M. et al. Nombres naturels. PERMAMA, (P184-3029), Télé-université, 1977.

BOYER, M. and M. MARISSE. Fonctions et calculs numériques à l'aide de la calculatrice. PERMAMA, (P184-3032 et P184-3042), Télé-université, 1979.

COLLETTE, J.P. Histoire des mathématiques. Paris: Editions du Renouveau pédagogique, 1973.

COTÉ, Benoît. Raisonnement mathématique et ordinateur. Télé-université, (PERMAMA), P184-4009, 1977.

CROWLEY, Thomas H. Understanding Computers. New York: McGraw-Hill, 1967.

CURTIS, Charles W. Linear Algebra, An Introduction Approach. Boston: Allyn and Bacon, Inc., 1967.

DALOIS, Maurice. APL: une approche pratique. Québec: Ministère de l'Éducation, 1980.

DIENES, Zoltan P. and Malcolm A. JEEVES. Thinking in Structures. Toronto: Hutchinson Educational, 1965.

DUFORT, J.G. et al. Résolution de problèmes, guide d'activités. PERMAMA, (P184-4002), Télé-université, 1977.

DUFORT, J.G. et al. Résolution de problèmes, introduction à l'heuristique. PERMAMA, (P184-4002), Télé-université, 1977.

ENGEL, A. A Short Course in Probability. Missouri: CEMREL, 1970.

FEHR, Howard F. Unified Mathematics. Menlo Park, Ca.: Addison-Wesley, 1972.

FELDMAN, Jérôme. "Les languages de programmation," in Pour la science. No. 28, February 1980, pp. 80 to 91.

FILION, P. Les entiers relatifs. Québec: S.o.e.m., n.d.

FLETCHER, T. L'apprentissage de la mathématique aujourd'hui. Paris: O.C.D.L. (Entreprises culturelles), n.d.

FLETCHER, T. L'algèbre linéaire et ses applications. N.p.: Cédic, n.d.

FLEURY, M. et al. Programmation linéaire 1. PERMAMA, (P184-3020), Télé-université, 1977.

FLEURY, M. et al. Programmation linéaire 1, textes supplémentaires. PERMAMA, (P184-3020), Télé-université, 1977.

FLEURY, M. et al. Programmation linéaire 2. PERMAMA, (P184-3021), Télé-université, 1976.

FOMIN, S.V. Number Systems. (Translated from Russian by Joan W. Teller and Thomas P. Branson). Chicago: University of Chicago Press, 1974.

GALION. Mathématique 4e, 5e, et 6e. Paris: O.C.D.L. (Hatier), n.d.

GOTTFRIED, Byron S. Schaum's Outline of Theory and Problems of Programming with Basic. New York: McGraw-Hill, 1975.

GRAVEL, H., P. PATENAUDE, and M. LABELLE. La mathématique en atelier. (Booklets 1 to 14). Paris: O.C.D.L. (Entreprises culturelles), n.d.

GUÉRIN, VASCHALDE, WARUSFEL. Le calculateur de poche et ses jeux. Paris: Hachette, 1976.

HAGELBARGER, DAVID and SAUL FINGERMAN. Cardiac: Cardboard Illustrative Aid to Computation. New York: n.p., 1969.

HARRIS, J. et al. Activités mathématiques. (Translation and adaptation by School Mathematics Project), Books A and B, by B. Bourget, P. Fillon, and C. Gaulin. Éd. Toronto: Holt, Rinehart and Winston, 1976.

Institut national de recherche et de documentation pédagogiques. Language symbolique d'enseignement LSE, manuel d'utilisation. Paris, January 1975.

JACOBS, Harold R. Mathematics, A Human Endeavor; A Textbook for Those Who Think They Don't Like the Subject. San Francisco, W.H. Freeman, 1970.

KAUFMAN, A. Des points et des flèches, la théorie des graphes. Paris: Science-poche, Dunod, n.d.

LOCKE, P. "Residue Designs," in The Mathematics Teacher. Reston, Virginia: NCTM, March 1972.

LOYD, Samuel. Mathematical Puzzles. New York: Dover Publications, 1959-1960, 2 vol.

National Council of Teachers of Mathematics. Historical Topics for the Mathematics Classroom. Reston, Virginia: NCTM, 1969.

NICHOLS, E. and E. Henry GARLAND. Modern Trigonometry. New York: Holt, Rinehart and Winston, 1968.

OGILVY, Charles S. and John T. ANDERSON. Excursions in Number Theory. New York: Oxford University Press, 1966.

OYSTEIN, Ore. Graphs and Their Uses. New York: Random House, 1963.

The Open University. Relations. M10019. London: BBC, n.d., 30 min. colour film.

PICARD, A. "Some Observations on Graphing in Modular Systems", in The Mathematics Teacher. Reston, Virginia: NCTM, May 1971.

SAWYER, W.W. Voir et comprendre les mathématiques. Montréal: Éditions Hurtubise HMH, 1972.

SCHLOSEBERG, S., and J. PROCHNAN. 50 jeux avec votre calculatrice de poche. Paris: Éditions Radio, n.d.

SMITH, David E. History of Mathematics. New York: Dover Publications, 1951-53, Vol. II.

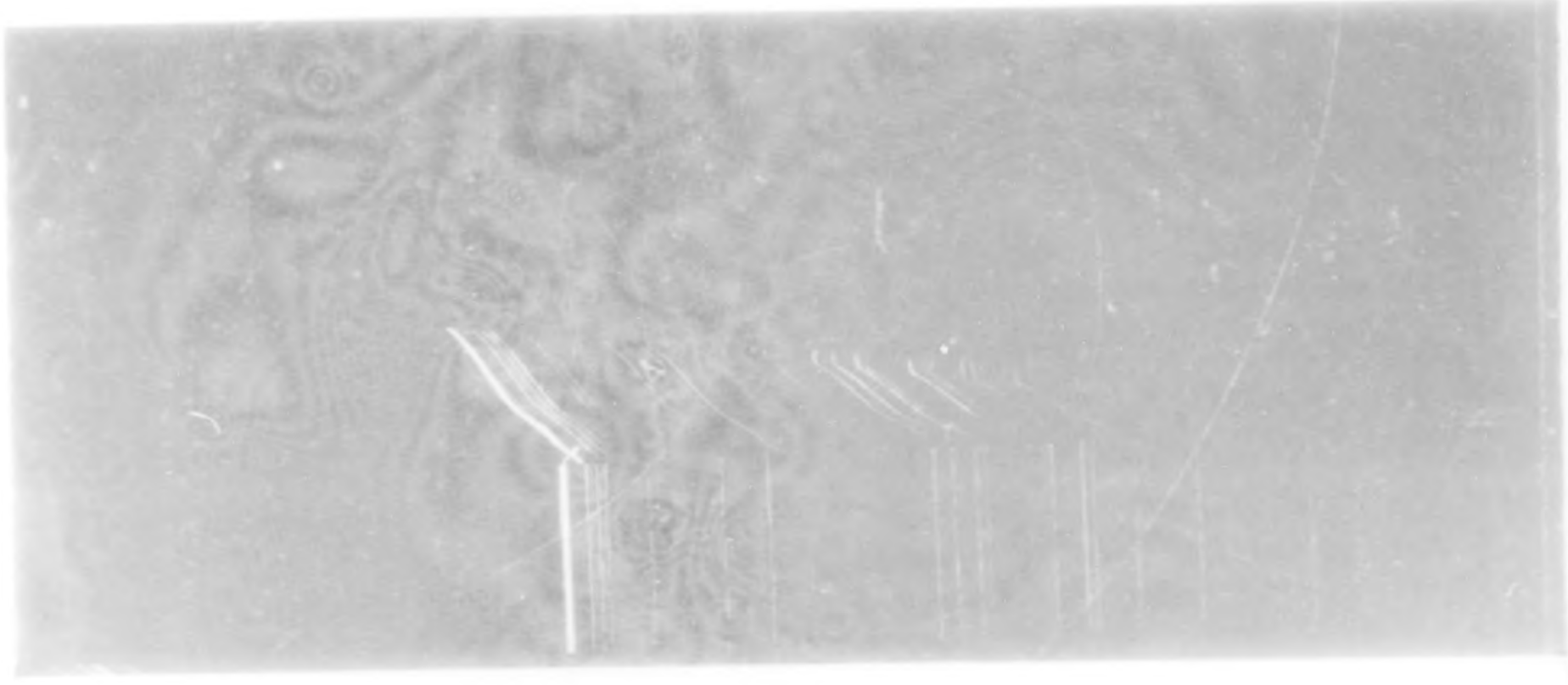
SPITZBARTH, Laurel M. Basic COBOL Programming: Self-instructional Manual and Text. Reading, MA: Addison-Wesley, 1970.

STEIN, Sherman K. Mathematics: The Man-made Universe. San Francisco: W.H. Freeman, 1969.

VANNIER, E. and P. CHAUVREAU. Comment jouer avec votre calculatrice de poche. Paris: Fayard, 1976.

VANNIER, E. Nouveaux jeux pour votre calculatrice de poche. Paris: Les Éditions Fayard, n.d.

WARUSFEL, André. Dictionnaire raisonné de mathématiques. Paris: Éditions du Seuil, 1966.

WARUSFEL, A. Les nombres et leurs mystères. Paris: Le Rayon de la Science, Édition du Seuil, 1961.

ZAKS, Rodney. Your First Computer: A Guide to Business and Personal Computing. Berkley: Sybex, 1980.